# TORSIONAL VIBRATIONS AND STABILITY OF THIN-WALLED BEAMS OF OPEN SECTION RESTING ON CONTINUOUS ELASTIC FOUNDATION

A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY IN MECHANICAL ENGINEERING

*By*

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DEDICATED

IN LOVING MEMORY

TO

Late Professor K. VENKATA APPA RAO

Head of the Department of Mechanical Engineering

Principal, College of Engineering

ANDHRA UNIVERSITY, WALTAIR.

## DECLARATION

I declare that this work is entirely original and has not been submitted in part or full for any Degree, Diploma or Title of any other University.

WALTAIR,
AUGUST, 1975.

(C. KAMESWARA RAO)
AUTHOR.

# CERTIFICATE

This is to certify that this thesis is the original work done by Sri Chellapilla Kameswara Rao, B.Sc., B.E., M.E., Associate Lecturer, Department of Mechanical Engineering, Andhra University, Waltair and that this thesis has not been submitted in part or full for any degree or diploma of any other University.

(Prof.P.K.SARMA)
Guide And Supervisor.

## BIOGRAPHICAL SKETCH.

The author was born in Peddapuram, East Godavary District, Andhra Pradesh, on 27th December 1947, and was the eldest son of Chellapilla Bangareswara Sarma, Head Master, Municipal High School, Narasaraopet, Guntur District, Andhra Pradesh. He was educated in Narasaraopet Municipal High School and S.S.N.College and obtained Bachelor of Science Degree from Andhra University in 1965. In 1968, he received the Bachelor's Degree in Mechanical Engineering from Sri Venkateswara University. He received the Master's Degree in Machine Design from Andhra University in February 1971.

He worked as Junior Research Fellow (C.S.I.R.) and Senior Research Fellow (U.G.C.) during 1970-'71 and joined the Faculty of Mechanical Engineering, Andhra University College of Engineering, Waltair in December 1971, as Associate Lecturer. He registered for Ph.D. in 1972 and worked under the guidance of Late Professor K.V.Appa Rao. Upon his sudden demise, he completed his Doctoral thesis work under the guidance of Prof.P.K.Sarma.

The author was married in May 1970, and has two sons, Hema Chandra Kumar and Suresh Babu.

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## NOMENCLATURE.

### Dimensions and Sectional Properties:

$A$ = total area of cross section

$A_f$ = area of each flange

$b_f$ = width of ~~the bar~~ each flange

$C_s$ = torsion constant

$C_w$ = warping constant

$F$ = constant depending upon cross sectional properties, see Eq.(10.4)

$h$ = height between the centerlines of the flanges

$I_f$ = moment of inertia of each flange about y-axis

$I_p$ = polar moment of inertia of the cross section

$I_R$ = fourth moment of inertia about the shear center, see Eq.(10.5)

$I_{pc}$ = half the polar moment of inertia about the shear center

$K'$ = numerical shape factor for cross section

$L$ = length of the beam

$t_f$ = thickness of each flange

$t_w$ = thickness of the web

$S_o$ = statical moment with respect to neutral axis

$z$ = displacement along the length of the bar

### Material Properties:

$E$ = Young's modulus

$E_{zz}$ = modulus for extension-compression along the axis of the bar

$G$ = shear modulus

$G_{zx}$ = shear modulus of orthotropic material

$K_t$ = foundation modulus in torsion

$\rho$ = mass density of the material of the beam

<u>Forces, displacements and Moments</u>:

$M$ = moment in each flange

$M_y$ = net bending moment in the cross section

$P$ = axial compressive load

$P_{cr}$ = torsional buckling load

$P^*$ = post-buckling load

$q$ = external viscous force per unit length acting along the sides of the flanges opposing warping

$Q$ = shear force due to bending in the flanges

$T_e$ = external torque per unit length of the beam

$T_o$ = a constant equal to the static torque

$T_s$ = torsional couple

$T_t$ = $T_s + T_w$ = total torque

$T_w$ = warping torque

$u$ = x-displacement of the top flange center line

$w$ = z-displacement of a point in the top flange

$\phi$ = angle of twist

$\bar{\phi}$ = normal function of $\phi$

$\phi_s$ = contribution of shear strain to the angle of twist

$\phi_t$ = angle of twist when shear strain has been neglected

$\psi$ = warping angle

$\bar{\psi}$ = normal function of $\psi$

Stresses and Strains:

$\sigma_x$, $\sigma_y$, $\sigma_z$ = normal stresses in x, y and z directions respectively

$\tau_{zx}$ = maximum shear stress in flange bending

$\epsilon_{sh}$ = shear strain at the center of the flange, x=0

$\epsilon_z$ = z-component of strain

Energies and Matrices:

$\bar{A}$ = transformation matrix for displacements whose elements are functions of x, y and z

$\bar{C}$ = transformation matrix giving the strains in terms of generalized displacements

$\bar{D}$ = matrix of material constants

$\bar{F}$ = total load matrix

$\bar{K}$ = total stiffness matrix

$\bar{M}$ = total mass matrix

$\bar{q}$, $\bar{R}$ = column matrices of generalized displacements

$\bar{Q}$, $\bar{r}$ = column vectors of amplitudes of generalized displacements

$\bar{S}$ = total stability coefficient matrix

$T_k$ = kinetic energy of the strained bar

u = components of the displacement vector

U = total strain energy

W = potential energy

$\bar{\sigma}$ = matrix of stresses

$\bar{\epsilon}$ = matrix of strains

Non-dimensional Parameters:

$\bar{a}^2$ = $1+s^2K^2 - K^2/\lambda^2 d^2$

$d^2$ = $I_f h^2/2 I_p L^2$ = longitudinal inertia parameter

$K^2$ = $GC_s L^2/EC_w$ = warping parameter

$r_n$ = ratios of eigen values (n=1 to 4)

$s^2$ = $EI_f/K' A_f GL^2$ = shear parameter

$\bar{t}_1$ = $(EC_w/\rho I_p L^4)^{1/2} t$ = dimensionless time

Z = z/L = non-dimensional beam length

$\bar{\alpha}_3$ = $E_{zz}/G_{zx}$

$\bar{\beta}_3$ = $(C_s + 1/2\, K' A_f h^2)/ I_p$

$\bar{\eta}$ = $K' A_f h^2/I_f$

$\bar{\xi}_2$ = $C_s/I_p$

$\Delta^2$ = $PI_p L^2/AEC_w$ = axial load parameter

$\Delta^2_{cr}$ = $P_{cr} I_p L^2/AEC_w$ = torsional buckling load parameter

$\Delta^{*2}_{cr}$ = $P^* I_p L^2/AEC_w$ = post-buckling load parameter

$\gamma^2$ = $K_t L^4/4EC_w$ = foundation parameter

$\lambda^2_c$ = $1/s^2 d^2$ = critical frequency parameter

$\lambda^2_n$ = $\rho I_p L^4 p_n^2/EC_w$ = frequency parameter

$\Omega^2$ = $2I_f/I_p$

$\delta^{*}$ = $F/C_w$

Miscellaneous:

$c_o$ = bar velocity = $(E_{zz}/\rho)^{1/2}$

$c_2$ = shear wave velocity = $(G_{zx}/\rho)^{1/2}$

$c_p$ = phase velocity for torsional waves

$i$ = $\sqrt{-1}$

$n$ = mode number

$N$ = Number of segments into which the beam is subdevided

$p_n$ = natural frequency of vibration in radious per unit time.

$t$ = time

$T$ = linear period of torsional vibration

$T^{*}$ = non-linear period of torsional vibration

$X$ = normal function giving the shape of mode of vibration

$\alpha_n$, $\alpha_n'$, $\beta_n$ = positive real quantities (n=1,2,3)

$\beta^{*}$ = torsional amplitude in non-linear analysis

$\beta_t$ = torsional damping constant

$\beta_w$ = warping damping constant

$\tau_n$ = torsional excitation function

$\tau^{*}$ = a function of time in non-linear analysis

$\epsilon^{*}$ = error function

$\delta$ = variational operator

$\delta_1$ = wave number = $2\pi/\lambda$

$\omega$ = torsional excitation frequency

$\lambda$ = wavelength

Salient symbols are listed above. Other symbols are defined in the body of the thesis as and when they appear.

## ABSTRACT

This thesis presents some analytical studies of linear and non-linear torsional vibrations and stability of uniform thin-walled beams of open section resting on continuous elastic foundation subjected to a time-invariant axial compressive load including the effects of longitudinal inertia and shear deformation.

Based on the Timoshenko torsion theory, the problem of linear torsional vibrations and stability of uniform lengthy thin-walled beams of open section resting on continuous elastic foundation subjected to a time-invariant axial compressive load is analyzed exactly by using the method of separation of variables. The frequency or buckling load and normal mode equations are derived for various end conditions. Approximate expressions are derived for the torsional frequency and buckling loads using Galerkin's technique. The results presented for some typical boundary conditions reveal that for lower modes, the increase in the foundation parameter increases the frequency parameter significantly and the increase in the axial load parameter decreases the frequency parameter considerably. The combined influence of axial load and foundation parameters is observed to be the superimposition of the individual effects on the frequency of vibration.

Finite element formulation of the problem of free torsional vibrations of thin-walled beams of open section resting on continuous elastic foundation is also presented. The stiffness and consistent mass matrices are derived and the eigen value problem

is formulated. The eigen values obtained by finite-element method compared favourably well with the exact values even for a coarse subdivision of the beam into six elements. A digital computer programme is written for obtaining the results for the frequency parameter for various boundary conditions.

As the corrections due to second order effects may be of importance if the effect of cross sectional dimensions on frequencies of vibration are desired, an exact analysis is presented for free torsional vibrations of short thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. New frequency and normal mode equations are derived for six common types of simple and finite beams. Solutions of the frequency equations for some typical boundary conditions are obtained on a digital computer. The individual effects of longitudinal inertia and shear deformation on the torsional frequencies of a simply supported beam are shown graphically. The torsional frequency values and the modifying quotients for the first four modes of vibration for some typical boundary conditions are presented in tabular form suitable for design use; they show the combined effects of longitudinal inertia and shear deformation. Approximate frequency equations for some typical end conditions are obtained using Galerkin's technique. It is observed that the effect of shear deformation is to decrease the stiffness of the beam and thus results in corresponding decrease of natural frequencies. The decrease is relatively small compared to the increase due to warping; however, the importance of shear deformation appears when higher frequencies are considered.

A finite-element formulation of the problem of free-torsional vibrations of short thin-walled beams of open section including the effects of longitudinal inertia and shear deformation is also presented. The corresponding stiffness and mass matrices including these second order effects are derived. The eigen values obtained by the finite element method compared very well with the exact values even for a coarse sub-division of the beam into three elements. A digital computer programme is written for obtaining the results for the frequencies and mode shapes for various end conditions.

The problem of forced torsional vibrations of thin-walled beams of open section is studied including the effects of longitudinal inertia and shear deformation. Viscous damping forces arising separately from torsional and warping velocities are included. The two coupled, fundamental equations of motion are formulated in terms of angle of twist and warping angle. The method of solution is demonstrated for arbitrary external torque for the beam having both ends simply-supported. Numerical results are presented for the case when the torque is uniform over the span and varies sinusoidally in time. Amplitude response is plotted against torsional excitation frequency for varying amounts of torsional and warping damping and is compared to the response for the classic beam for the first five symmetric mode shapes. The amplitudes for the thin-walled beam including shear deformation and longitudinal inertia are found to be considerably larger.

As the increased utilization of composite materials in structural applications has made their analysis ever more important, the problem of torsional wave propagation in orthotropic thin-walled beams of open section including longitudinal inertia and shear deformation is solved. The equation for free torsional vibrations of thin-walled beams of open section of orthotropic material including the effects of longitudinal inertia and shear deformation is established analogous to that for isotropic materials. Many fiber-reinforced plastics and pyrolytic-graphite type materials which are mostly in use, are orthotropic or transversely isotropic in the sense that the ratio of in-plane modulus of elasticity to shear modulus is large. It is shown that, for these materials, the corrections due to longitudinal inertia and shear deformation may be of one order of magnitude greater than the corrections in the isotropic case. Graphs are given of the phase velocity versus inverse wavelength for various aspect ratios of beams of different materials.

The problem of torsional vibrations and stability of short thin-walled beams of open section resting on continuous elastic foundation and subjected to an axial compressive load including the effects of longitudinal inertia and shear deformation is solved by means of an exact analysis. Results for buckling loads for various boundary conditions are presented in tabular form showing the effects of shear deformation. The values of torsional frequency parameter for the first four modes of vibration for various boundary conditions and non-dimensional parameters are presented in tabular form suitable for design use. This problem is also solved by means of finite-element method and an excellent agreement is observed between the results from exact analysis and those from the finite-element method.

It is very well known that a large number of problems of torsional vibrations and stability of thin-walled beams arising in modern high speed aircraft structures, missiles and launching vehicles cannot be adequately explained by the classical linear theories alone, since the torsional deformations of these beams are usually of such a magnitude that the assumption of small rotations of cross sections will no longer be valid.

In view of this, an attempt has been made further in this thesis to derive and solve the governing differential equation of large amplitude torsional stability of lengthy thin-walled beams of open section resting on continuous elastic foundation. Graphs indicating the combined influence of large amplitude and foundation parameter on the torsional post-buckling loads for simply supported and clamped beams are presented. Including the effects of axial compressive load and elastic foundation, the problem of non-linear torsional vibration and post-buckling behavior of thin-walled beams resting on continuous elastic foundation is also investigated.

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# CHAPTER - I

# INTRODUCTION.

## 1.1. GENERAL:

In an effort to save weight, while still retaining high strength capabilities, many contemporary structural systems are designed with lower margins of safety than their predecessors. The criterion of minimum weight design is particularly prevalent in the design of aircraft, missile, and space craft vehicles. One obvious means of obtaining a high strength, minimum weight design is the use of light, thin-walled structural members of high strength alloys. For intricate structures such as space-crafts, beams of standard cross section may not be the most efficient or convenient structural members to use. Thin-walled beams of open section are frequently employed for their structural efficiency. With the improvement of extrusion methods in metal forming, beams of different shapes of cross sections can be formed to order. Occasions often arise when uniform doubly symmetric cross sections are more convenient to use. Examples of such structural members that have gained great favour as stiffeners in aerospace design are the I,Z, Channel and angle sections.

Although no attempt has been made in the previous paragraph to regorously define a thin-walled beam, it is necessary to do so in order that one fully understands its meaning when used in ensuing discussion. A rectangular beam as a structural member is characterized by having two dimensions, the width and depth of the cross section of comparable size but small in comparision with the third dimension, the length. A thin-walled beam, on the other hand, is characterized by its three dimensions being of different orders of magnitude. The thickness of the beam is small compared to the characterestic dimensions of the cross section, and the cross sectional dimensions are small compared to the length of the beam.

It has long beam known that a beam with nonsymmetrical cross section under loads will, in general, not only deflects but also will also twist. Only under special loading along the flexure axis, a line joining the shear centers, will the beam deflect without twist. The concept of shear center is well known and is discussed in text books. Essentially, it is a point through which the resultant of the shear forces of the cross section passes. If the loading does not pass through the shear center, a torque is generated by the loading and the resultant of the reactions from the section. Such a torque will cause the twisting of the beam. When a thin-walled beam is subjected to dynamic excitation, the inertial loading due to acceleration of the beam itself has to be taken into account. The resultant of such loading may be considered to pass through the centroid of the section. Unless the shear center of the section coincides with its centroid, both bending and torsional vibrations will result. Due to the low torsional rigidity of thin-walled open section beams, the problem of torsional vibrations and stability is of primary interest.

## 1.2. BRIEF REVIEW OF RELEVANT LITERATURE:

Extensive research has been conducted in the field of thin-walled structural members which has been well documented in the literature, and detailed bibliographies are already available. Therefore, only a brief survey of the development of the existing literature directly related to the present investigation will be included here.

### 1.2.1. ELASTIC STABILITY:

Since the eighteenth century investigation of column instability by Euler, a great wealth of information has been documented concerning the nature of instability. For instance, the instability of columns, beam-columns, plane frames, trusses, plates, and shells have been the objects of many research efforts. Although the indifidual investigations are too numerous to cite, several texts have appeared that provide excellent anthologies for these investigations.

Derivation of the fundamental theory of strength and stability of thin-walled members was performed by Goodier, Timoshenko, Vlasov and others. Timoshenko ( 98) initiated the concept of non-uniform torsion when he considered warping of the cross sections of a symmetrical I-beam subjected to torsional moment. Wagner ( 110 ) generalized the Timoshenko torsion theory. Goodier (36,37) published a series of studies in which he simplified and proved some of the assumptions proposed by earlier investigators. Theories of lateral stability and flexural-torsional stability of uniform thin-walled beams, upto 1945, were unified by Timoshenko ( 98 ). Vlasov's (101) extensive investigations of thin-walled elastic members were published in book form in 1940. A new edition containing comprehensive study of equilibrium, stability, and Vibration of thin-walled members of arbitrary cross sections was published in Russian in 1958 and translated into English in 1961.

Two other classical text books dealing with the stability of members were published by Bleich ( 13 ) in 1952 and Timoshenko and Gere ( 99) in 1961. Most recent is Ziegler's monograph (114), in 1968, on structural stability in which he emphasizes the conceptual aspects of the more recent developments of stability theory. Surveys of the theory of thin-walled members, which include numerous references, were performed by Nowisinki ( 87) in 1959, Panovko ( 89) in 1957 and Yi-Yuan, Yu ( 113) in 1971. A survey of literature on the lateral instability of beams was made in 1960 by Lee ( 73 ). The effect of axial stresses, arising from combined bending and torsion of thin-walled beams, on the torsional regidity of the beam was investigated by Goodier (38 ) in 1951 and Engel (29 ) in 1953.

In 1944, Goodier and Barton extended Timoshenko's theory of non-uniform torsion of an I-beam to include not only the bending of the flanges in their own planes but also considered the effect of web deformation on the torsion of the beam (~~15~~ 39). Further investigation of this effect including experimental work was performed by several researchers. The Goodier-Barton effect was found to be of significant importance for the case of plate girders whose cross sections were such that the ratio of the flange thickness to the web thickness was large or if the length of the web was much larger than the length of the flange (35,71).

Gregory (42) in 1961, proposed a theory which considered a non-linear longitudinal stress system in members subjected to large elastic torsional displacements. Gregory's theory was developed by Black (11,12) in 1965 and in 1967, in a theoretical and experimental study of monosymmetric thin-walled beams subjected to bending and torsion. Approximate solutions of a modified non-linear equation were compared with the experimental results and also with the theories of Timoshenko (98) and Goodier (38). A continuous effort has also been made to close the gap between structural theory and engineering codes of practice (5,18,12). Recent research studies of interest to designs and research workers are presented in a collection of papers, published in 1967, on the stability and strength of thin-walled structural members and frames (16).

The influence of second order effects such as distortion of the column cross section, large displacements, shear deformation, residual stress and initial deflections on the behaviour of biaxially loaded columns ~~is~~ was evaluated by Culver (22) in 1965. Numerical calculations, including these second order effects, indicated that problems exist for which these effects are considerable. Second order effects influencing biaxially loaded columns were discussed by Goodier ( 40 ) and Heilig ( 44 ) and these effects included cross sectional distortion due to torsion and shear deformations.

Tapered thin-walled beams are of interest in optimum design. Gere and Carter ( 33 ) obtained the critical buckling loads for tapered columns. A finite element formulation using Gelerkin's method for the buckling problem of tapered members was presented by Morrel and Lee ( 82 ). The elastic stability of axially loaded tapered columns has been studied analytically by several investigators (27,84). The problem of torsional buckling of axially loaded tapered columns of wide-flanged cross section has been recently studied analytically by Culver and Preg ( 23 ), using finite-difference method. In addition, the differential equations for the general case of tapered wide-flanged beam-columns have been derived using the Vlasov's method (107 ) for uniform beams. The determination of the initial yield load for tapered beam-columns has also been investigated ( 30 ). An experimental investigation of the elastic stability of tapered beam-columns has been reported (15 ). Lee ( 74 ) presented an analysis of non-uniform torsion of tapered I-beams in 1956, the taper being only of a restrictive type.

All the above investigations and a host of others treat the torsional or flexural - torsional buckling problems from a purely mathematical approach. Such an approach includes the solution of a trio of coupled differential equations of equilibrium (these equations may be uncoupled under some instances) for columns of various cross sections, loadings and boundary conditions. This approach provides one with exact solutions (mathematically speaking) for a given problem. One shortcoming of such an approach is that due to the complex nature of the equilibrium equations such mathematical difficulties as non-uniform members, complex loadings, or arbitrary boundary conditions can not be easily handled.

To complement the known exact solutions, attempts have been made to obtain approximate solutions to the more difficult (again, mathematically speaking) problems. The technique used to obtain the approximate results is the method of finite or discrete element technique. Many of the early advances in the finite element method were presented in technical journals, but recently texts by Przemieniecki ( 93 ) and Zienkiewicz (115), have appeared that summarized various investigations utilizing this modern technique. These texts cover such varied topics as plane stress, plane strain, axisymmetric stress analysis, three dimensional stress analysis, bending of beams, plates and shells, vibrations of elastic systems, and structural stability.

Using the finite-element technique, Krajcinovic ( 68 ) developed a formulation for thin-walled members based on the use of hyperbolic functions to express the twist. These functions, which are the solution to the exact differential equation for twist, lead to complicated stiffness expressions in torsional and warping constants. It does not include the effects of instabilities due to torques. Hence, its applicability to general frame instability is limited. Kabaila and Fraeijsde Venbeke( 46) formulated a finite-element model that considers only axial forces in the stability analysis. The formulation is only applicable to solid beams where the shear center coincides with the center of gravity. It neglects warping rigidity, which is of major importance in the analysis of thin-walled members( 98 ). A linear formulation was used to express the twist, as was done earlier, by Przemieniecki ( 93 ). The finite-element method has been shown, by Pardoen ( 90 ), Barsoum ( 6, 8 ) and Barsoum and Gallangher ( 7 ) to be completely general in that it provides one with a means of solving problems involving arbitrary loading and boundary conditions. Although, only an approximate method, the finite-element method has provided results that are sufficiently accurate for engineering purposes.

1.2.2. VIBRATIONS AND WAVE-PROPAGATION:

For the past three decades mechanical vibrations have been recognized as a major factor in the design of air craft, marine and machine structures. Mechanical vibrations produce increased stress, energy loss and noise that should be considered in the design stages if these undesirable effects are to be avoided, or kept to a minimum. This is essentially true in the area where the total mass of the system is to be held to a minimum. Vibratory motion can produce very disastrous results as in the case of either the Tacoma narrows bridge which fell because the wind excited it at a natural frequency, or the ill-fated Electra I Commercial air craft that encountered severe engine vibration which required major modification of air craft.

The important point to be noted is that too often vibrations are investigated after, instead of before, the failure has occured.

Several investigators have been concerned with the vibration of beams and the purpose herein is to review some of the relevant contributions in this area. The most desirable technique for analyzing vibratory motion is the regorous mathematical solution obtained from a formal solution of the differential equations describing the motion. Timoshenko and Young (100) investigated the coupled torsional and transverse vibrations of a simply supported beam having a constant channel cross section. He considered only the simply supported beam and by assuming a product form solution was able to obtain an algebraic frequency equation. This technique is limited to only those cases in which it is possible to assume a solution for the mode shape that satisfies the physical constraints of the beam. Gere (32) studied the torsional vibration of beams with doubly-symmetric cross section for which the shear center and centroid coincide and analyzed the effect of warping on the frequencies of torsional vibration and the shapes of the normal modes of vibration for bars of single span with various end conditions. Gere and Lin (34) generalized the theory of vibrations of thin-walled beams of arbitrary open section.

The above cited references presented classical mathematical solutions for the beam vibration problems. Wherever possible the use of these formal mathematical solutions is highly recommended because they are the simplest and most direct methods of predicting vibratory characterestics. However, it should be noted that each of these formal solutions has very definite limitations because they have been obtained for a specific type of beam and are not applicable to the general case. Since there had not been developed a rigorous mathematical technique that will solve all types of beam vibration problems, it was only natural that various approximate techniques have been developed to fill in the gaps left in the formal solutions. One of the most powerful techniques developed was the Rayleigh-Ritz method which is an energy principle that in the absence of frictional losses, the total vibratory energy of a vibrating body must continuously change from all strain energy and no kinetic energy to all kinetic energy and no strain energy, and the frequency of change must be a natural frequency.

The first step in the application of the Rayleigh-Ritz method is to assume a possible model shape of the beam corresponding to the lowest frequency. Then it will be possible to calculate the maximum strain energy in the beam. By considering that the assumed mode shape is periodic in time the maximum kinetic energy can be obtained. When the two energies are equated, it is possible to solve for the frequency. Succeeding possible mode shapes must be assumed until the lowest calculated frequency is obtained. This technique converges only to the lowest natural frequency of the system. The higher natural frequencies can be obtained only by using the orthogonality property that exists between the mode shapes. A complete discussion of the Rayleigh-Ritz technique is presented by Temple and Beckley ( 96 ).

Garland ( 31 ) used the Rayleigh-Ritz method to investigate the coupled torsional and transverse vibration of cantilever beams having constant channel cross section. He was able to observe that for any one transverse mode of vibration there will be two torsional modes and that the coupled natural frequency can be expressed as functions of the uncoupled transverse and uncoupled torsional frequencies. Timoshenko ( 100 ) was also able to make this observation for a simply supported channel cross-section. Garland was able to obtain a remarkable degree of correlation between the predicted and the experimentally measured results. Because he was dealing with only the lowest natural frequencies, he was not in requirement of the use of the orthogonality condition that would be necessary for obtaining the higher natural frequencies.

Bennett ( 9 ) developed an improved matrix technique for investigating the vibratory characterestics of a beam having a plane of symmetry perpendicular to the plane of transverse vibration. For a beam having a non-collinear longitudinal mass and shear center axes, there will be a coupling between the transverse and torsional vibrations. The coupling is produced when the reversed effective force caused by the transverse vibration does not act through the shear center of the cross-section. To date there has not been developed a rigorous mathematical solution for all possible variations in cross section, loading conditions and methods of support. Several authors have solved the equations by imposing specific limitations on the method of support or on the variation of the cross section. Some researchers have used an energy method or an iterative method to approximate solutions where the formal solution does not exist. These approximate methods have a tendency to become very tedious. The technique of investigating the higher natural frequencies introduces complexities that are difficult to understand physically. The matrix method proposed by Bennett ( 9 ) is valid for any loading conditions or method of support. In his work, three different types of beam vibrations are considered, coupled torsional and transverse, transverse alone and torsional alone. The governing differential equations were solved approximately by using a digital computer and results obtained are observed to be within the range of engineering accuracy.

Another approximate but more elegant technique is the finite-element technique which provides one with solutions for any general set of boundary conditions and the variation in the cross section. This technique has been successfully used by Mei (77,78) for the solution of the coupled bending-torsion vibrations of thin-walled beams of open section and non-linear flexural vibrations of rectangular beams. Pardoen ( 90 ) and Barsoum ( 6 ) presented satisfactory solutions for the vibration and dynamic stability problems of thin-walled beams of open section utilizing the finite-element method. Although the finite-element technique has been used to predict the natural frequencies and mode shapes of beams, the method has yet to be extended to consider the torsional vibrations and stability of thin-walled beams of open section resting on continuous elastic foundation.

Stress wave propagation in elastic solid media have been subjected to analysis since the early investigations of poisson ( 92). Recent developments have been motivated by the ever increasing need for information concerning the response of structures to high dynamic loads. The beam as a fundamental element of structures, received the first attention of investigators in the field. The early work of Pochhammer ( 91 ) and Chree ( 17 ) on the circular cylindrical bar with traction-free surface was re-examined in the early 1940's but progress was slow on account of highly intricate transcendental frequency equations resulting from dispersion due to the presence of boundaries. The first three modes of longitudinal and flexural wave transmission were not known until found by Davies ( 24) in 1948 and Abramson( 1 ) in 1957.

The complexity of the exact analysis even for simple geometry of a circular cylindrical bar, emphasized the need for physically satisfactory approximate theories. To satisfy engineering requirements, these theories should be good for short wave lengths which occur in problems of steep transients or high frequency oscillations in bars. The elementary classical theories of Navier for longitudinal vibrations, Bernoulli-Euler for flexural vibrations and Coulomb for torsional oscillations were reviewed and with the exception of the latter, were found to lead to physically impossible results ( 72 ). As a consequence, emphasis was placed on developing more accurate approximate theories for longitudinal and flexural vibrations.

Although Timoshenko ( 101 ) in 1921 proposed a theory for flexural oscillations which included the effects of shear deformation and rotary inertia, it was not until the last decade that the Timoshenko theory was really put to experimental and analytical tests. During this period, in addition to a lot of allied literature on exact theories of plates, and over a dozen of books, monographs and surveys, not less than fifty papers appeared dealing with approximate theories. These papers included new theories, their mutual comparison, comparision with the known information from exact theories and experiment. The Timoshenko theory for flexural waves and the Mindlin-Herrmann theory ( 81 ) for longitudinal waves were found most satisfactory. The rest of the literature with the propagation of pulses is based on these theories. Brief details have been previously summarized by Kolsky ( 67 ), Abrahmson, Plass and Ripperger( 2 ), Green ( 41 ), and more recently by Redwood ( 89 ) and Miklowitz( 80

However, comparable torsional oscillation analysis was virtually neglected and not more than four to five papers on the topic have been published. The reason is the fact that Coulomb classical theory gives the same first-mode results as the exact theory. The available information is almost limited to the circular cylindrical bar. Thus, there exists a lack of satisfactory approximate and exact theories for torsional wave propagation in non-circular bars, especially these used in structural applications. Very often thin-walled beams of open section are used as structural members in light weight aircraft and building construction. These members usually fail under torsion or combined bending torsion because of their low torsionaly rigidity which makes them susceptible to torsional buckling. A self-contained and comprehensive account of bending and torsion of thin-walled beams of open section was given in a paper published by Timoshenko (98) in 1945. As structural members may be subjected to resonant vibrations under dynamic loads, it is necessary to study their torsional properties in order to understand their response to torsional excitation.

The inadequacy of a Saint-Venant elementary torsion theory for short wave lengths was hinted at by Love ( 76 ), who suggested a correction for the longitudinal inertia associated with torsional deflection. However, both the elementary theory and Love's approximation have the same defects as do their counterparts in longitudinal wave-propagation theory. The dynamic equation used by Gere (32 ) in his torsion analysis was essentially that previously derived by Timoshenko( 98) and he studied the effect of warping of the cross-section on the frequencies of vibration. These equations constitute the Timoshenko Torsion theory in the sequel and are found to lead to physically absurd results for short wave length waves.

To present a much needed practical engineering theory, a strength of materials theory was derived and analyzed by Aggarwal ( 3 ) in his thesis, including the effects of shear deformation, longitudinal inertia and warping of the cross-section. At high frequencies and short wave lengths a new mode of the wave transmission is added. This arises from the coupled interaction of the torsional deformation and bending effects of shear deformation and longitudinal inertia. The Aggarwal's theory lead to theoretically satisfactory results for the first mode of transmission over a wave length spectrum which included moderately short wave lengths, and agrees with previous approximations for large wave lengths. The group velocity for the second mode is shown to increase monotonically from zero for the longest waves to the bar velocity for very short wave lengths, which is in agreement in form with the higher modes of the exact theory for circular cylindrical bars (88, 25). In many respects the analysis of Aggarwal's theory proves to be analogous to that of Timoshenko's flexural theory (101).

The transient response arising from a step torque applied impulsively at the end of a semi-infinite I-beam is analyzed by Aggarwal (3) and the non-dimensional equations ~~are~~ were solved using Laplace transforms and a closed form solution in integral form is obtained. For the sake of comparison, he solved the same impulsively applied step torque problem according to the Timoshenko torsion theory. He also analyzed the problem of free and forced vibrations of I-beams according to his theory which includes the effects of longitudinal inertia and shear deformation. He noticed a completely new spectrum of natural frequencies at higher frequencies due to the interaction between torsion, shear deformation and longitudinal inertia effects The frequency equations and expressions for model functions are derived for a number of cases but he limited the discussion regarding the existence of the second frequency spectrum only to the case of the simply supported beam because of the highly transcendental nature of the frequency equations which further include the parameters of warping, shear and longitudinal inertia. The frequencies obtained according to his theory are were compared with those previously obtained by Gere ( 32 ) who used the Timoshenko torsion equation. The shear effect is shown to result in a decrease of beam stiffness and corresponding decrease of natural frequencies. Though, the decrease is were relatively small compared to the increase due to warping; the influence of shear deformation were is observed to be considerable at higher frequencies. Further, Aggarwal ( 3 ) established an Orthogenality relation for the principal modes of vibration and treated the problem of forced vibrations under very general loa

Where as Aggarwal's contribution was limited to an improvement of the previous theories of uncoupled torsional vibrations, Tso's contribution (104) was in the field of coupled torsional and bending vibrations of thin-walled beams of open section. In his thesis, Tso (104) derived a higher order theo including the effect of shear strain induced by bending and warping of the beam. He compared the spectrum curves of the highe order theory with those from the elementary theory for various boundary conditions for a special family of non-symmetric sections. He performed an experiment on two specimens to determin their natural frequencies at different beam lengths and compared the experimental results with those predicted from the two theories. He has concluded that when the beam is long, the elementary theory is adequate to predict the natural frequencies for torsion predomenant modes. For bending predominant modes, the higher order theory should be used. The higher order theory derived by Tso ( 104 ) serves also as a guided for the range of validity of the elementary theory. In the experimental observations, he found certain non-linear behaviour of the thin-walled beam. Under special circumstances, when the beam is excited at resonance at a higher mode, he observed a tendency for the beam to shift from the higher resonant mode to vibrate at its fundamental mode, resulting in a higher order subhormonic oscillation. Hence he made an analysis to show the possibility of such a behaviour if the inherently non-linear governing equations for coupled torsional and bending vibrations are used.

Recently In 1967, Aggarwal and Cranch ( 4 ) published a paper as an extension to the work of Aggarwal ( 3 ), by including an analysis for the coupled bending-torsional vibrations of a channel beam. The equations governing the motion of the channel beam are derived using Hamilton's principle and include the effects of warping, longitudinal inertia and shear deformation. These equations explicitly resemble those derived by Tso (104) for the more general case of mono-symmetric thin-walled beam of open cross section. However, the approach of Aggarwal and Cranch seems to be different from that of Tso. Whereas Tso, analyzed the vibrations of a monosymmetric thin-walled beam, torsional wave analysis is made by Aggarwal and Cranch for the case of an I-beam and a channel beam.

A more general theory of vibrations of cylindrical tubes which includes the secondary effects such as transverse shear, longitudinal inertia and shear lag was presented by Krishnamurthy and Joga Rao ( 70 ). They also brought out the analogy between the flexural and torsional vibrations of doubly symmetric tubes. In Part IV of their theory ( 70 ), results for simply supported open tube of doubly symmetric I - section were presented. The other boundary conditions were not analyzed.

## 1.3. AIM AND SCOPE OF THE PRESENT INVESTIGATION:

In the above investigations (34, 70, 104) on the torsional vibrations of thin-walled beams of open section including the second order effects such as longitudinal inertia and shear deformation, only regorous mathematical solutions are attempted. This approach actually limited their solutions only to simple end conditions such as a simply supported beam. Stating that, the frequency equations are highly transcendental in nature, Aggarwal ( 3 ) did not attempt the solutions for boundary conditions other than the simply supported ends. However, with the advent of high speed digital computers, it is not too difficult to obtain the solutions for these transcendental frequency equations.

The present thesis aims at developing exact and approximate methods of analysis to tackle various boundary conditions without much difficulty. An attempt has been made, to extend the previous discussions on torsional vibrations and stability analysis of thin-walled beams of open section, to include the effects of axial compressive load, continuous elastic foundation, longitudinal inertia and shear deformation by making use of exact and approximate methods of analysis. A non-linear analysis is also made to study the influence of large torsional amplitude on the non-linear period of vibration. Further, the effects of axial compressive load and continuous elastic foundation on non-linear torsional behaviour of thin-walled beams of open section are also investigated.

In particular, Chapter II deals with the analysis of torsional vibrations and stability of lengthy uniform thin-walled beams of open section resting on continuous elastic foundation and subjected to a time-invarient axial compressive load by means of exact and approximate methods. A finite-element formulation for the same problem which is useful both for uniform and non-uniform beams is presented in Chapter III. The comparison between the results from the exact analysis and approximate finite element method is shown to be excellent even for a coarse sub-division of the beam.

In Chapter IV, an exact analysis is presented for free torsional vibrations of short uniform thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. Expressions for orthogonality and normalizing conditions for the principal normal modes which are useful in solving forced vibration problems and which include both the angle of twist and warping angle are obtained for both the general case and for beams with various simple end conditions. To fescilitate the designers, extensive design data pertaining to wide-flanged I-beams with various end conditions is presented. Also, approximate frequency equations for clamped and clamped-simply supported beams are derived making use of the Galerkin technique. A finite element formulation of the problem is presented in Chapter V. New stiffness and mass matrices are presented which includes the effects of longitudinal inertia and shear deformation. The results obtained by the finite element method are in good agreement with the exact ones.

An analysis for the forced torsional vibrations of thin-walled beams of open section including the effects of longitudinal inertia, shear deformation and viscous damping is given in Chapter VI. Chapter VII deals with the problem of torsional wave propagation in orthotropic thin-walled beams of open section including the effects of longitudinal inertia and shear deformation.

In Chapter VIII, the problem of torsional vibrations and stability of short uniform thin-walled beams resting on continuous elastic foundation and subjected to an axial static compressive load including the effects of longitudinal inertia and shear deformation is analyzed by means of an exact method. Approximate expressions for the frequency and buckling load are derived for clamped and clamped-simply supported beams utilizing Galerkin's technique. A finite-element solution of the same problem is presented in Chapter IX.

A non-linear analysis for the torsional stability of thin-walled beams of open section at large amplitudes is presented in Chapter X. In Chapter XI, the effects of axial time-invariant compressive load and elastic foundation on the non-linear torsional vibrations and stability are analyzed. In Chapter XII, salient conclusions are arrived at, bringing out the practical significance of the problems solved. Also the scope for further investigation is discussed.

Available reprints of the papers published on part of the work presented in this thesis are enclosed at the end for ready reference. The rest of the material is accepted for publication in reputed Journals and is awaiting publication.

# CHAPTER - II

## TORSIONAL VIBRATIONS AND STABILITY OF LENGTHY THIN-WALLED BEAMS ON ELASTIC FOUNDATION - EXACT AND APPROXIMATE ANALYTICAL SOLUTIONS.*

### 2.1 INTRODUCTION:

Static and dynamic analysis of beams on elastic foundation occupies a prominant place in contemporary structural mechanics. The vibrations and buckling of continuously supported finite and infinite beams resting on elastic foundation has an application in the design of highway pavements, aircraft runways and in the use of metal rails for rail road tracks. A very large number of studies have been devoted to this subject, and valuable practical methods for the analysis of beams on elastic foundation have been worked out.

Regarding the static analysis of beams on elastic foundation Hatenyi's book (43) ~~is rather a~~ classic giving the complete development of the beams supported on elastic foundation. A later development of the theory ~~is~~ beautifully presented by Vlasov and Leovitiv (108) in their book on ''beams, plates, and shells on elastic foundation'' with improved models of elastic foundation. Since the actual response at the interface depends on the material of the foundation and is usually very difficult to determine, various foundation models were proposed to approximate the real foundation behavior among which Winkler's constant modulus foundation is widely used because of its simplicity. A discussion of various foundation models is presented by Kerr (65).

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* Part of the results from this chapter were published by the author and A.A.Satyam in February 1975 issue of AIAA Journal, see Ref. 47.

The effect of shear flexibility is included in the analysis of beams on elastic foundation by Ractliffee ( 94). Biot ( 10 ) treated the bending of an infinite beam on elastic foundation and Conway and Farmham ( 19 ) analyzed the bending of a finite beam in bonded and unbonded contact with an elastic foundation. Recently Niyogi ( 86 ) presented an approximate analysis of axially constrained beam on elastic foundation and Murthy ( 83 ) solved the problem of buckling of continuously supported beams. The problem of buckling of thin-walled beams of open section such as I-beams, channel sections etc., with continuous elastic supports has been treated by Timoshenko and Gere ( 97) in their book on ''Theory of elastic stability''. By using the finite element method, Pardoen ( 90 ) analyzed the buckling of thin-walled beams of open section resting on continuous elastic supports subjected to an axial load.

On the dynamics side of beams on elastic foundation, Kenney ( 66 ) analyzed the steady state flexural vibrations of beams on elastic foundation for a moving load including the effect of viscous damping. Crandall (20 ) analyzed the flexural vibrations of a beam on elastic foundation including the effects of rotary inertia and shear deformation. Tseitlin (103) determined the effects of shear deformation and of rotary inertia in flexural vibrations on beams on elastic foundation. Lloyd and Miklowitz (75) presented an analysis for the flexural wave propagation of beams and plates on an elastic foundation.

While there exists a good number of investigations on flexural vibrations of rectangular beams or plates on elastic foundation, the literature on the torsional vibrations of beams on elastic foundation is rather scarce. To the best of authors knowledge the effects of a time-invarient axial compressive load and of elastic foundation on the torsional frequency and buckling loads of thin-walled beams of open section are not being analyzed anywhere in the available literature. To this end, the present chapter deals with the exact and approximate analytical solutions of the effects of a time-invariant axial compressive load and of elastic foundation on the torsional frequency and buckling loads of lengthy thin-walled beams of open section.

## 2.2. BASIC ASSUMPTIONS:

The problem investigated in this chapter is restricted to the following assumptions:

a) The thin-walled beam has uniform open cross sections along its length.

b) Strains are assumed to remain within the elastic limit. The curvature and twist of the beam are considered to be small. In particular, the deformations are small compared with the cross-sectional dimensions of the beam in the linearized problem.

c) The beam is fabricated from material which is homogeneous and isotropic and which obeys Hooke's law ( a linearly elastic material).

d) The centroid and shear center of the cross section coincide.

e) Shearing strains of the middle surface due to shear and warping effects, and axial strains of the beam due to longitudinal load components are considered to be negligibly small (the beam is undergoing inextensional motions).

(f) Longitudinal inertia effects are considered to be negligibly small. Conditions (e) and (f) are referred to as the Timoshenko Torsion theory.

(g) Distortion of the cross sections in their own planes is not considered, however, warping of the sections is permitted. Distortion of the sections would be of significance for built-up girders or if the cross section is very deep or very wide.

(h) No internal or external damping forces are considered.

## 2.3 DERIVATION OF BASIC DIFFERENTIAL EQUATION:

As the cross sectional dimensions are assumed to be small compared to the length of the beam, the second order effects such as longitudinal inertia and shear deformation can be treated as negligible.

In this section, based on Timoshenko torsion theory ( 98 ), the governing differential equation of free motion of a doubly symmetric thin-walled beam on elastic foundation subjected to a time-invariant axial compressive load is derived utilizing Hamilton's principle. The method has the advantage of generating the natural boundary conditions which shall be discussed in section 2.4.

Hamilton's principle (87), states that for dynamical process:

$$\delta \int_{t_0}^{t_1} ( T_k - U + W ) \, dt = 0 \qquad (2.1)$$

where (T - U + W) is the Lagrangian function, $T_k$ the kinetic energy of the strained bar, U the total strain energy, W the potential energy of the external force, and $t_0$, $t_1$ are two fixed instants.

Fig.1.1 shows a differential element of length dz of a wide-flanged I-beam undergoing torsion. According to Saint Venant, the cross-sections are assumed to rotate about the centroid-shear center 'O' giving rise to a torsional couple,

$$T_s = GC_s \frac{\partial \phi}{\partial z} \tag{2.2a}$$

where G is the shear modulus, $C_s$ the torsion constant for the cross section, and $\phi$ (z, t) the angle of twist.

The torsion constant for an I-section is given by

$$C_s = (2bt_f^3 + ht_w^3)/3 \tag{2.2b}$$

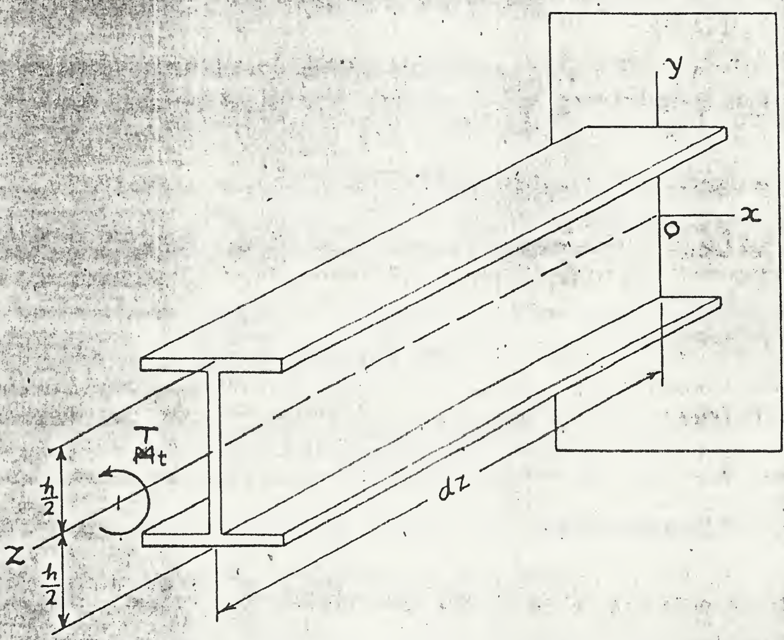
where b is the width of the flanges, h the height between the centerlines of the flanges, $t_f$ the thickness of the flanges, and $t_w$ the thickness of the web.

The strain energy $U_1$ at any instant t in the beam of length L due to Saint Venant torsion is

$$U_1 = \frac{1}{2} \int_0^L GC_s \left( \frac{\partial \phi}{\partial z} \right)^2 dz \tag{2.2c}$$

Accompanying the rotation is a warping of the section which is assumed constant in each piece of the cross section hav- a moment M. The x-displacement of the top flange centerline, u

FIG. 2·1 – DIFFERENTIAL ELEMENT OF A WIDE-FLANGED I-BEAM



is given by

$$u = (h/2)\,\phi \qquad (2.2d)$$

and hence the moment M in the top flange is given by

$$M = EI_f \frac{\partial^2 u}{\partial z^2} = EI_f \frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} \qquad (2.2e)$$

where E is the Young's modulus, $I_f$ the moment of inertia of each flange area about y-axis.

It can be easily observed that the moment M in the top flange and -M in the bottom flange cancel so that no net moment $M_y$ exists in the cross section.

The shear force Q due to the bending of the flanges is given by

$$Q = \frac{\partial M}{\partial z} = EI_f \frac{h}{2} \frac{\partial^3 \phi}{\partial z^3} \qquad (2.2f)$$

The equal and opposite shear forces Q, a distance h apart in the top and bottom flanges, give rise to a torque due to warping, $T_w$, given by

$$T_w = -\,Qh = -\,EI_f \frac{h^2}{2} \frac{\partial^3 \phi}{\partial z^3} = -\,EC_w \frac{\partial^3 \phi}{\partial z^3} \qquad (2.2g)$$

where $C_w = I_f\,h^2/2$ is the warping constant for an I-section (32).

The total torque, $T_t$, on the cross section is given by

$$T_t = T_s + T_w = GC_s \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} \qquad (2.2h)$$

If $U_2$ is the strain energy of the two flanges due to warping, then

$$U_2 = \frac{1}{2} \int_0^L 2 \, EI_f \left(\frac{\partial^2 u}{\partial z^2}\right)^2 dz = \frac{1}{2} \int_0^L EC_w \left(\frac{\partial^2 \phi}{\partial z^2}\right)^2 dz \qquad (2.2i)$$

The strain energy $U_3$ due to the Winkler type elastic foundation, is given by

$$U_3 = \frac{1}{2} \int_0^L K_t (\phi)^2 \, dz \qquad (2.2j)$$

Hence, the total strain energy U, at any instant t becomes

$$U = U_1 + U_2 + U_3 = \frac{1}{2} \int_0^L \left[ GC_s \left(\frac{\partial \phi}{\partial z}\right)^2 + EC_w \left(\frac{\partial^2 \phi}{\partial z^2}\right)^2 + K_t (\phi)^2 \right] dz \qquad (2.2)$$

The kinetic energy of rotation of the cross section at the corresponding instant is given as:

$$T = \frac{1}{2} \int_0^L \rho I_p \left(\frac{\partial \phi}{\partial t}\right)^2 dz \qquad (2.3)$$

where $I_p$ is the polar moment of inertia of the cross section and $\rho$ the mass density of the material of the beam.

The potential energy due to the external time-invariant axial compressive load, P, acting at the centroid of the cross section at the corresponding instant is given by

$$W = \frac{1}{2} \int_0^L \frac{PI_p}{A} \left(\frac{\partial \phi}{\partial z}\right)^2 dz \qquad (2.4)$$

where A is the area of the cross section.

Substituting for $T_K$, U and W from equations (2.2) to (2.4) respectively in equation (2.1), taking the variations of the integrand, and integrating the first term by parts with respect to t and the next four terms with respect to z, one obtains:

$$\int_{t_o}^{t_1}\int_{o}^{L}\left\{\left|(GC_s - \frac{PI_p}{A})\frac{\partial^2\phi}{\partial z^2} - EC_w\frac{\partial^4\phi}{\partial z^4} - K_t\phi - \rho I_p\frac{\partial^2\phi}{\partial t^2}\right|\delta\phi\right\} dz\, dt$$

$$+ \int_{o}^{L}\rho\, I_p\frac{\partial\phi}{\partial t}\,\delta\phi\,\Big|_{t_o}^{t_1} dz - \int_{t_o}^{t_1} EC_w\frac{\partial^2\phi}{\partial z^2}\,\delta\left(\frac{\partial\phi}{\partial z}\right)\Big|_{o}^{L} dt$$

$$- \int_{t_o}^{t_1}\left\{(GC_s - \frac{PI_p}{A})\frac{\partial\phi}{\partial z} - EC_w\frac{\partial^3\phi}{\partial z^3}\right\}\delta\phi\,\Big|_{o}^{L} dt = 0 \qquad (2.5)$$

Assuming that the values of $\phi$ are given at the two fixed instants, the second integral vanishes. If the boundary conditions are such that the third and the fourth integrals also vanish, then the associated differential equation of motion is given by:

$$(GC_s - \frac{PI_p}{A})\frac{\partial^2\phi}{\partial z^2} - EC_w\frac{\partial^4\phi}{\partial z^4} - K_t\,\phi - \rho I_p\frac{\partial^2\phi}{\partial t^2} = 0 \qquad (2.6)$$

2.4 (a) NATURAL BOUNDARY CONDITIONS:

In deriving the basic differential equation of motion (2.6) from (2.5) it was assumed that the expressions

$$EC_w\frac{\partial^2\phi}{\partial z^2}\quad\delta\left(\frac{\partial\phi}{\partial z}\right) \qquad ($$

and

$$\left[ (GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} \right] \delta\phi$$

vanish at the ends z = 0 and z = L. These conditions are satisfied if at the two ends

$$\frac{\partial^2 \phi}{\partial z^2} \delta (\frac{\partial \phi}{\partial z}) = 0, \qquad (2.7)$$

and

$$\left[ (GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} \right] \delta\phi = 0 \qquad (2.8)$$

Equation (2.7) and (2.8) give the natural boundary conditions for the finite bar, and are satisfied if the end conditions are taken as

(1) $\phi = 0$ and $\frac{\partial^2 \phi}{\partial z^2} = 0$ (2.9)

These conditions imply restraint against rotation but not against warping; that is, the end of the bar does not rotate but is free to warp. This is the case of a ''Simple Support''.

(2) $\phi = 0$ and $\frac{\partial \phi}{\partial z} = 0$ (2.10)

These conditions imply restraint not only against rotation but also against any warping of the end cross section. This means that the end of the bar is built-in rigidly so that no deformation of the end cross section can take place. These conditions define a ''Fixed Support''.

(3) $\frac{\partial^2 \phi}{\partial z^2} = 0$ and $(GC_s - \frac{PI_p}{A})\frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} = 0$ (2.11)

These conditions imply no restraint of any kind at the end of the bar. This requires that the bending moment in the flange ends and torque acting on the end cross section must be zero. These conditions correspond to a ''free end''.

(4) $\frac{\partial \phi}{\partial z} = 0$ and $(GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} = 0$

or equivalently

$$\frac{\partial \phi}{\partial z} = 0 \text{ and } \frac{\partial^3 \phi}{\partial z^3} = 0 \qquad (2.12)$$

The latter conditions imply no warping and zero shear forces in the end flanges.

These conditions are useful for finding symmetric modes of vibration in simply supported, fixed-fixed and free-free beams.

(b) TIME-DEPENDENT BOUNDARY CONDITIONS:

The homogeneous boundary conditions discussed above, give the free vibrations of bars. For forced vibrations produced by the motion of boundaries, appropriate time dependent end conditions are given by prescribing at each end one member of each of the products:

$$EC_w \frac{\partial^2 \phi}{\partial z^2} \bar{\delta}(\frac{\partial \phi}{\partial z}) \text{ and } \left| (GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} \right| \bar{\delta}\phi$$

or equivalently of:

$$M \bar{\delta}(\frac{\partial \phi}{\partial z}) \text{ and } T_t \bar{\delta}\phi.$$

Of the many conditions thus obtained, the following are of more theoretical interest:

1. Twisting moment $T_t$ prescribed, flange bending moment $M = 0$ or $\frac{\partial \phi}{\partial z} = 0$,

2. $\phi$ or $\frac{\partial \phi}{\partial t}$ prescribed, flange bending moment $M = 0$ or $\frac{\partial \phi}{\partial z} = 0$,

3. Flange bending moment M prescribed, twisting moment $T_t = 0$ or $\phi = 0$,

4. $\frac{\partial \phi}{\partial z}$ or $\frac{\partial^2 \phi}{\partial z \partial t}$ prescribed, twisting moment $T_t = 0$ or $\phi = 0$.

In the case of semi-infinite beams, conditions need be prescribed at one end since all physical quantities at any instant are zero at the far end.

2.5 ANALYSIS OF VARIOUS TERMS:

i) If $K_t = P = 0$ and $C_w = 0$, Eq.(2.6) reduces to

$$GC_s \frac{\partial^2 \phi}{\partial z^2} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (2.13)$$

This equation represents Saint Venant's torsion theory for slender beams and does not include warping of the cross-section shear deformation and or longitudinal inertia effects. It is given by Love (76) and is discussed by Gere (32).

ii) If $K_t = P = 0$, Eq.(2.6) reduces to

$$GC_s \frac{\partial^2 \phi}{\partial z^2} - EC_w \frac{\partial^3 \phi}{\partial z^3} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (2.14)$$

This equation represents Timoshenko's torsion theory which includes the effect of warping of the cross section and has been treated in detail by Gere (32).

(iii) If $K_t = 0$, Eq.(2.6) reduces to

$$(GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} - EC_w \frac{\partial^4 \phi}{\partial z^4} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (2.15)$$

This equation represents the effect of an axial time-invariant compressive load added to Timoshenko's torsion theory.

(iv) If P = 0, Eq.(2.6) reduces to

$$GC_s \frac{\partial^2 \phi}{\partial z^2} - EC_w \frac{\partial^4 \phi}{\partial z^4} - K_t \phi - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (2.16)$$

This equation represents the effect of Winkler type constant modulus elastic foundation added to Timoshenko Torsion theory.

## 2.6 NON-DIMENSIONALIZATION AND GENERAL SOLUTION OF EQUATION OF MOTION:

For mathematical simplification, it is convenient to to reduce Eq.(2.6) to a non-dimensional form, simultaneously introducing some dimensionless parameters having physical interpretations.

Introducing, $Z = z/L$, the non-dimensional beam length, and $\tau_1 = (\frac{EC_w}{\rho I_p L^4})^{1/2} t$, the dimensionless time variable, Eq.(2.6) in non-dimensionless form can be written as:

$$\frac{\partial^4 \phi}{\partial Z^4} - (K^2 - \Lambda^2) \frac{\partial^2 \phi}{\partial Z^2} + 4\gamma^2 \phi + \frac{\partial^2 \phi}{\partial \tau_1^2} = 0 \qquad (2.17)$$

where

$$K^2 = \frac{GC_s L^2}{EC_w} \text{, warping regidity parameter,} \qquad (2.18)$$

$$\Delta^2 = \frac{PI_pL^2}{AEC_w}\text{, axial load parameter,} \qquad (2.19)$$

and

$$\gamma^2 = \frac{K_tL^4}{4EC_w}\text{, foundation parameter,} \qquad (2.20)$$

The general solution of Eq.(2.17) can be obtained by using the standard method of separation of variables. Thus, by taking $\phi$ in the form

$$\phi = X\,(Z)\,T\,(t_1) \qquad (2.21)$$

and then substituting into Eq.(2.17), separating the variables, and setting the resulting expressions equal to $-\lambda_n^{\,2}$, we obtain

$$T = A_n \cos\lambda_n \bar{t}_1 + B_n \sin {}_n \bar{t}_1 \qquad (2.22)$$

The expression for a normal mode of vibration is then

$$\phi = X\,(A_n \cos\lambda_n \bar{t}_1 + B_n \sin\lambda_n \bar{t}_1) \qquad (2.23)$$

in which X is the normal function giving the shape of the mode of vibration and $\lambda_n$ is the dimensionless torsional frequency parameter given by

$$\lambda_n^2 = \frac{\rho I_p L^4 p_n^2}{EC_w}, \qquad (2.24)$$

Where $p_n$ is the natural frequency of vibration in radious per unit of time. Any actual motion of the vibrating beam can be obtained by a summation of normal modes, so that in the general case

$$\phi = \sum_{n=1}^{\infty} X_n (A_n \cos \lambda_n \bar{t}_1 + B_n \sin \lambda_n \bar{t}_1) \tag{2.25}$$

in which the coefficients $A_n$ and $B_n$ are found from the initial conditions of the vibration.

The equation for determining the normal function X, found by substituting Eq.(2.24) into the differential Eq.(2.17), is then

$$\frac{d^4X}{dZ^4} - (K^2 - \Delta^2) \frac{d^2X}{dZ^2} + (4\gamma^2 - \lambda_n^2) X = 0 \tag{2.26}$$

The general solution of this equation may be found by taking the normal function X in the form:

$$X = D' e^{\bar{\eta} Z} , \tag{2.27}$$

which yields the auxiliary algebraic equation:

$$\bar{\eta}^4 - (K^2 - \Delta^2) \bar{\eta}^2 + (4\gamma^2 - \lambda_n^2) = 0 \tag{2.28}$$

The four roots of the equation are

$$\bar{\eta}_1 = +\alpha_1, \quad \bar{\eta}_2 = -\alpha_1; \quad \bar{\eta}_3 = +i\beta_1, \quad \bar{\eta}_4 = -i\beta_1 \tag{2.29}$$

in which $\alpha_1$ and $\beta_1$ are the positive, real quantities given by

$$\alpha_1 = (1/\sqrt{2}) \left\{ (K^2 - \Delta^2) + \left[ (K^2 - \Delta^2)^2 + 4(\lambda_n^2 - 4\gamma^2) \right]^{1/2} \right\}^{1/2} \tag{2.30}$$

and

$$\beta_1 = (1/\sqrt{2}) \left\{ -(K^2 - \Delta^2) + \left[ (K^2 - \Delta^2)^2 + 4(\lambda_n^2 - 4\gamma^2) \right]^{1/2} \right\}^{1/2} \tag{2.31}$$

The general solution of Eq.(2.26) then becomes either

$$X = D_1' e^{+\alpha_1 Z} + D_2' e^{-\alpha_1 Z} + D_3' e^{+i\beta_1 Z} + D_4' e^{-i\beta_1 Z}$$

or

$$X = D_1 \cosh \alpha_1 Z + D_2 \sin h\alpha_1 Z + D_3 \cos \beta_1 Z + D_4 \sin \beta_1 Z \quad (2.32)$$

There are four orbitrary constants in this expression which must be determined so as to satisfy the particular boundary conditions of the problem. For any beam there will be two boundary conditions at each end and these four conditions determine the frequency equation and the ratios of three of the constants to the fourth constant. Solving the frequency equation then determines the principal frequencies of vibration. With the frequencies and normal functions determined, the solution is essentially complete.

## 2.7 FREQUENCY EQUATIONS AND MODEL FUNCTIONS:

In this section, frequency equations and mode shapes for some special cases are are established. Gere's results (32) are obtained for the special case $\Delta^2 = \gamma^2 = 0$. Because of the complexity of the frequency equations, the discussion of the results is limited to the case of simply supported beam.

BOUNDARY CONDITIONS: In section (2.4a) natural boundary conditions were discussed. By combining these conditions in pairs, many types of single-span beams can be analyzed. In terms of non-dimensional parameters, the boundary conditions can be written as:

1. Simple Support:

$$X = 0, \quad \frac{d^2X}{dZ^2} = 0 \qquad (2.33)$$

2. Fixed Support:

$$X = 0, \quad \frac{dX}{dZ} = 0 \qquad (2.34)$$

3. Free End:

$$\frac{d^2X}{dZ^2} = 0, \quad (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \tag{2.35}$$

Before we proceed to derive the frequency and Normal mode equations for various cases, from Equations (2.30) and (2.31) we obtain:

$$\alpha_1^2 = (K^2 - \Delta^2) + \beta_1^2 \tag{2.36}$$

and

$$\lambda_n^2 = \alpha_1^2 \beta_1^2 + 4\gamma^2 \tag{2.37}$$

If in case, the beam is not vibrating and only elastic torsional buckling is to be investigated the expressions for $\alpha_1$ and $\beta_1$ from Equations (2.30) and (2.31) reduce to:

$$\alpha_1 = (1/\sqrt{2})\left\{(K^2 - \Delta^2) + \left[(K^2 - \Delta^2)^2 - 16\gamma^2\right]^{1/2}\right\}^{1/2} \tag{2.38}$$

and

$$\beta_1 = (1/\sqrt{2})\left\{-(K^2 - \Delta^2) + \left|(K^2 - \Delta^2)^2 - 16\gamma^2\right|^{1/2}\right\}^{1/2} \tag{2.39}$$

The following frequency equations which we derive for various cases are also useful in finding the torsional buckling loads when the reduced Equations (2.38) and (2.39) are used for $\alpha_1$ and $\beta_1$ respectively. In this case the following relations are to be used:

$$\alpha_1^2 = -4\gamma^2/\beta_1^2 \tag{2.40}$$

and

$$\Delta^2 = K^2 + \beta_1^2 - \alpha_1^2 \tag{2.41}$$

### 2.7.1 SIMPLY SUPPORTED BEAM:

This is the simplest case which admits complete analytical treatment. An example is a beam supported by framing angle connections at the two ends. These beams are used in building construction and therefore are of practical importance.

The boundary conditions from Equations (2.33) are:

$$X = d^2X/dZ^2 = 0 \quad \text{at } Z = 0$$

and

$$X = d^2X/dZ^2 = 0 \quad \text{at } Z = 1$$

For the conditions at Z = 0, Equation (2.32) gives:

$$D_3 + D_1 = 0,$$

and $D_1(\alpha_1^2 + \beta_1^2) = 0.$

Since the secular determinant $\alpha_1^2 + \beta_1^2 \neq 0$, it follows that

$$D_1 = D_3 = 0 \qquad (2.42)$$

From the second pair of conditions, Equation (2.32) gives:

$$D_2 \sinh \alpha_1 + D_4 \sin \beta_1 = 0, \qquad (2.43)$$

and

$$D_2 \alpha^2 \sinh \alpha_1 - D_4 \beta^2 \sin \beta_1 = 0 \qquad (2.44)$$

For a non-trivial solution, the secular determinant must vanish. This gives the characterestic equation

$$(\alpha_1^2 + \beta_1^2) \sinh \alpha_1 \sin \beta_1 = 0$$

Since $\alpha_1^2 + \beta_1^2 \neq 0$, and sinh $\alpha_1 \neq 0$, we obtain the frequency equation for this case as:

$$\sin \beta_1 = 0 \qquad (2.45)$$

From Equation (2.45) we have,

$$\beta_1 = n\pi, \; n = 1,2,3,...... \qquad (2.46)$$

This is the frequency equation for a simply supported beam and by using the relations (2.36) and (2.37), we find the expression for the frequency parameter $\lambda_n$ as:

$$\lambda_n = \left| n^2\pi^2(n^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2 \right|^{1/2} \qquad (2.47)$$

Since $\sin \beta_1 = 0$, we find from Equation (2.43) or (2.44) that $D_2 = 0$. Hence the model function is

$$X = D_4 \sin n\pi Z \qquad (2.48)$$

The complete expression for the angle of twist $\emptyset$ is obtained by summing up the normal modes, so that

$$\emptyset = \sum_{n=1}^{\infty} \sin n\pi Z(A_n \cos \lambda_n t_1 + B_n \sin \lambda_n t_1)$$

in which $A_n$ and $B_n$ are determined by the initial conditions.

Gere (32) studied the influence of warping parameter K, and concluded that it increased the frequency of vibration as warping increases the stiffness of the bar against rotation. For small values of K, which means $C_w$ is relatively large, the effect of warping is considerable and must be taken into account. For large K, which means $C_w$ is relatively small, the warping effect is also small and may be neglected in many cases.

To estimate the individual influences of axial load and elastic foundation, Equation (2.47) can be reduced in the following manner.

(a) If the effect of axial load alone is to be studied, by putting $\gamma = 0$, we obtain

$$\lambda_1 = n\pi(n^2\pi^2 + K^2 - \Delta^2)^{1/2} \tag{2.49}$$

(b) If the influence of elastic foundation alone is to be investigated, by putting $\Delta = 0$, we get

$$\lambda_2 = \left[ n^2\pi^2 \, (n^2\pi^2 + K^2) + 4\gamma^2 \right]^{1/2} \tag{2.50}$$

(c) If the both the effects of axial load and elastic foundation are to be neglected, by putting $\Delta = 0$ and $\gamma = 0$, we obtain the equation that was derived by Gere (32) as:

$$\lambda_3 = n\pi \, (n^2\pi^2 + K^2)^{1/2} \tag{2.51}$$

Denoting by $r_1$ the ratio of the frequency of vibration with axial load alone considered, Equation (2.49), to the frequency with axial load also neglected, Equation (2.51), we obtain

$$r_1 = \frac{\lambda_1}{\lambda_3} = \left[ 1 - \frac{\Delta^2}{n^2\pi^2 + K^2} \right]^{1/2} \tag{2.52}$$

Similarly, denoting by $r_2$ the ratio of the frequency of vibration with elastic foundation alone considered, Equation (2.50), to the frequency with elastic foundation also neglected, Equation (2.51), we obtain

$$r_2 = \frac{\lambda_2}{\lambda_3} = \left| 1 + \frac{4\gamma^2}{n^2\pi^2(n^2\pi^2 + K^2)} \right|^{1/2} \tag{2.53}$$

To find the combined influence of axial load and elastic foundation, let us denote by $r_3$ the ratio of the frequency of vibration with both axial load and elastic foundation considered, Equation (2.47), to the frequency with both axial load and elastic foundation neglected, we obtain

$$r_3 = \frac{\lambda_n}{\lambda_3} = \left| 1 + \frac{4\gamma^2 - n^2\pi^2\Delta^2}{n^2\pi^2(n^2\pi^2 + K^2)} \right|^{1/2} \tag{2.54}$$

Fig.2.2 shows the variation of $r_1$ with $\Delta$, for values of K = 0.1, 1.0 and 10.0 for the first fundamental mode of vibration. The effect of axial load is to decrease the frequency of vibration, since the axial load decreases the stiffness of the bar against rotation. For small $\Delta$, which means axial load P is relatively small, the effect of axial load is small and for large $\Delta$, which means P is relatively large, the effect of axial load is quite considerable.

Figs.2.3 and 2.4 show the variation of $r_2$ with $\gamma$, for values of K = 1 and 10 respectively, for the first three modes of vibration. The effect of the elastic foundation is to increase the

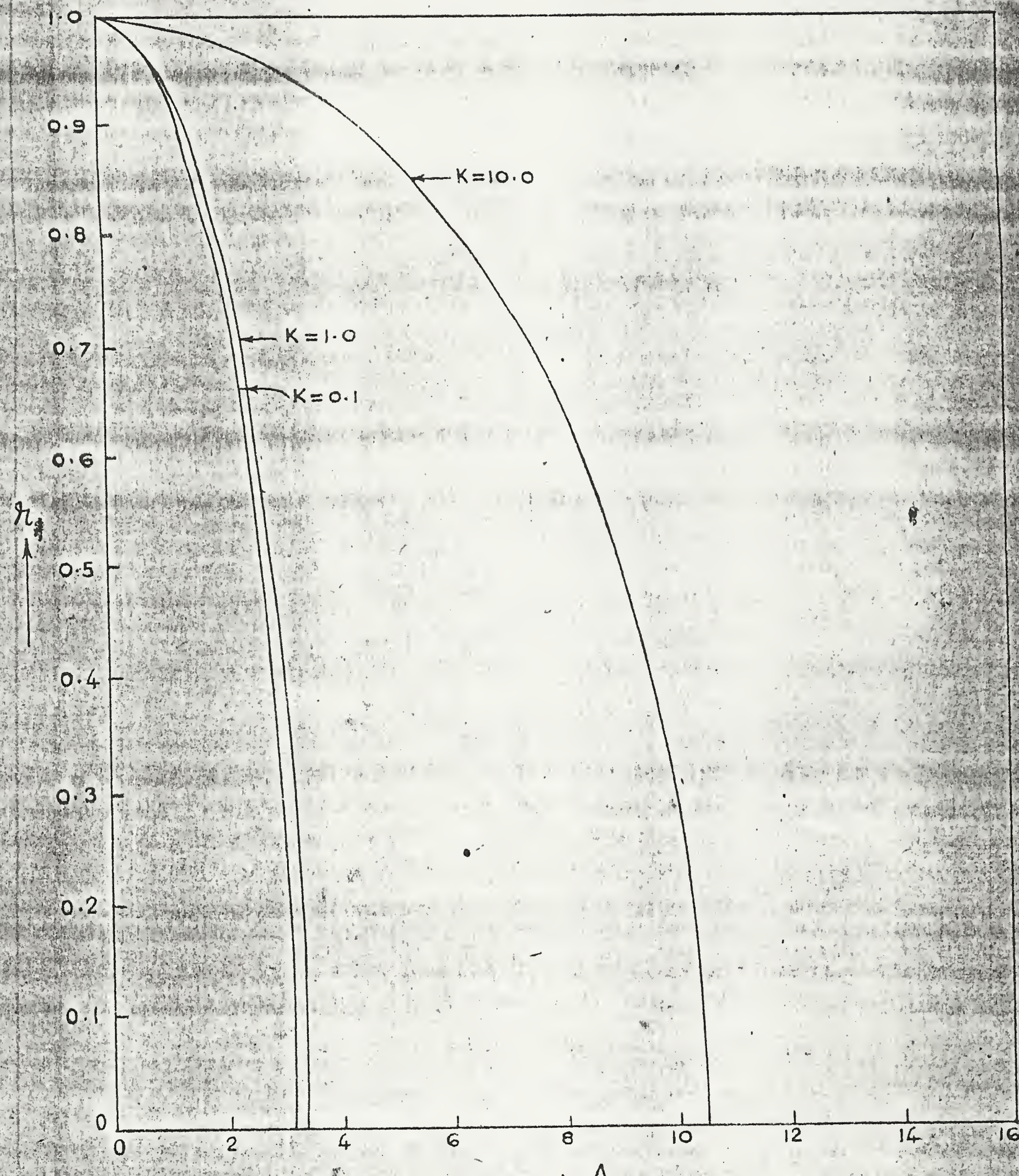



Fig. 2.2. Variation of $r_1$ with $\Delta$, for Values of K=0.1, 1.0 and 10.0 (n=1)

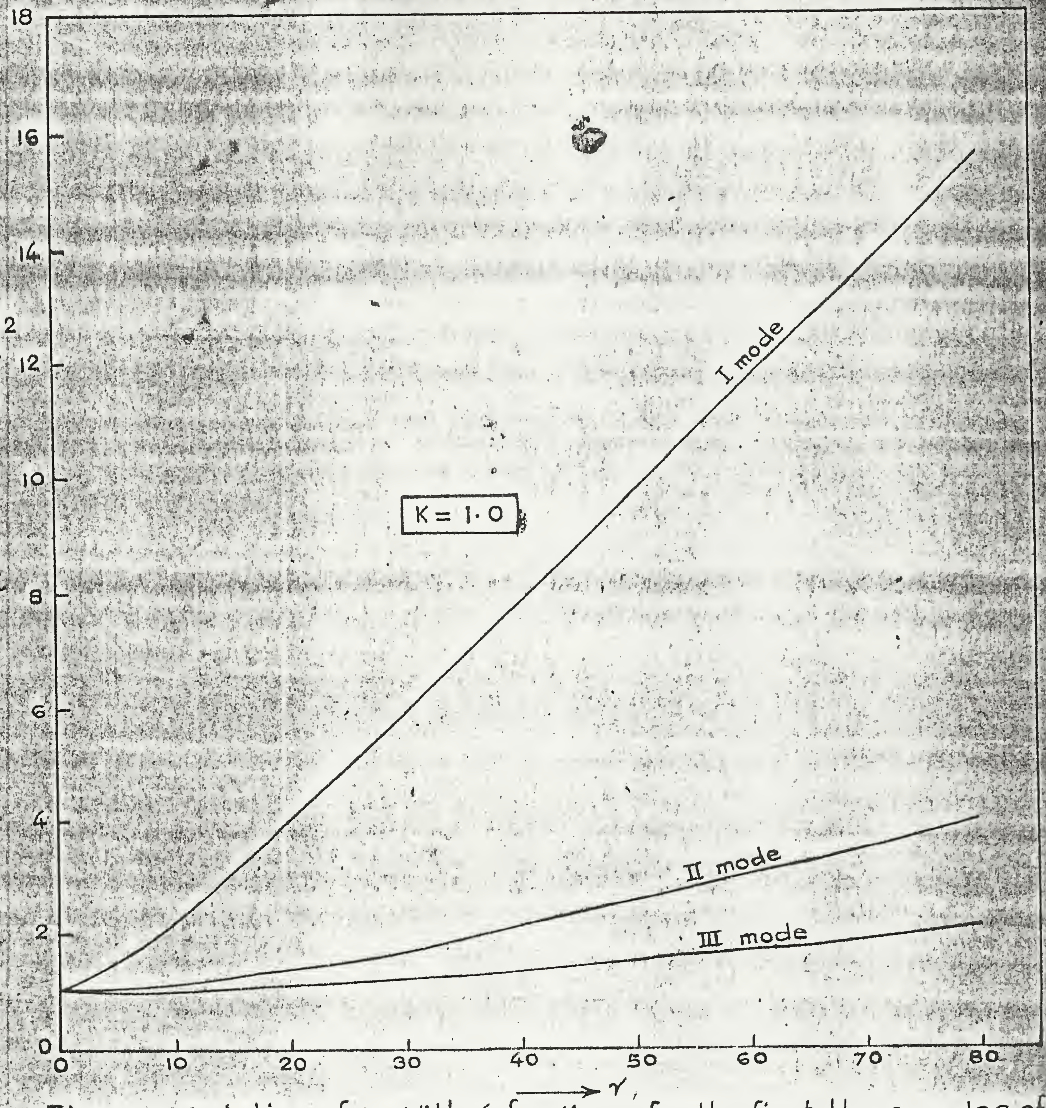



Fig. 2·3. Variation of $r_2$ With $\gamma$ for K=1·0 for the first three modes of Vibration.

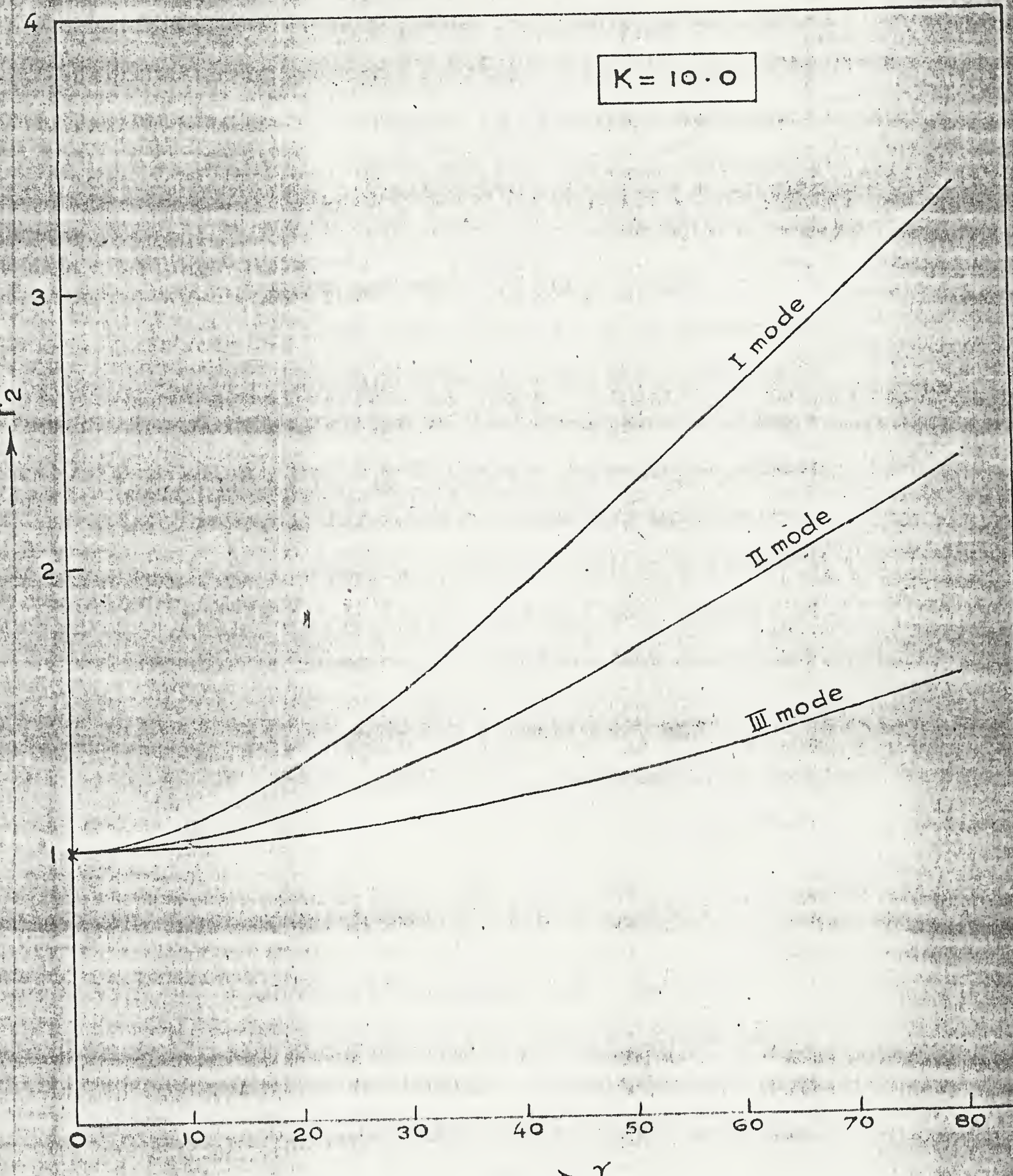



Fig. 2.4. Variation of $r_2$ with $\gamma$, for K=10·0 for the first three modes of vibration.

frequency of vibration, as the elastic foundation increases the stiffness of the bar against rotation. For small $\gamma$, which means foundation modulus $K_t$ is relatively small, the effect of elastic foundation is small and for large $\gamma$, which means $K_t$ is relatively large, the effect of elastic foundation is quite considerable.

Figs.2.5 and 2.6 show the variation of $r_3$ with $\Delta$ and $\gamma$, for values of K = 1 and 10, for the first fundamental mode of vibration. The combined effect of axial load and elastic foundation is the algebraic sum of individual influences which are actually opposite in nature. For a value of $\gamma^2 = 0.25\, n^2\pi^2 \Delta^2$, the combined influence of the axial compressive load and elastic foundation on the torsional frequency becomes zero. It can also be noticed from Equation (2.53) that the influence of elastic foundation decreases for higher modes of vibration.

When the beam is not vibrating, ie., $\lambda = 0$, we obtain from Equation (2.47), the expression for torsional buckling load (n=1) as,

$$\Delta^2_{cr} = \pi^2 + K^2 + (4/\pi^2)\gamma^2 \tag{2.55}$$

To show the influence of elastic foundation on the torsional buckling load, let us define by $r_4$, the ratio of the buckling load when elastic foundation is considered, to the buckling load when elastic foundation is neglected.

$$r_4 = 1 + \frac{4\gamma^2}{\pi^2(\pi^2 + K^2)} \tag{2.56}$$

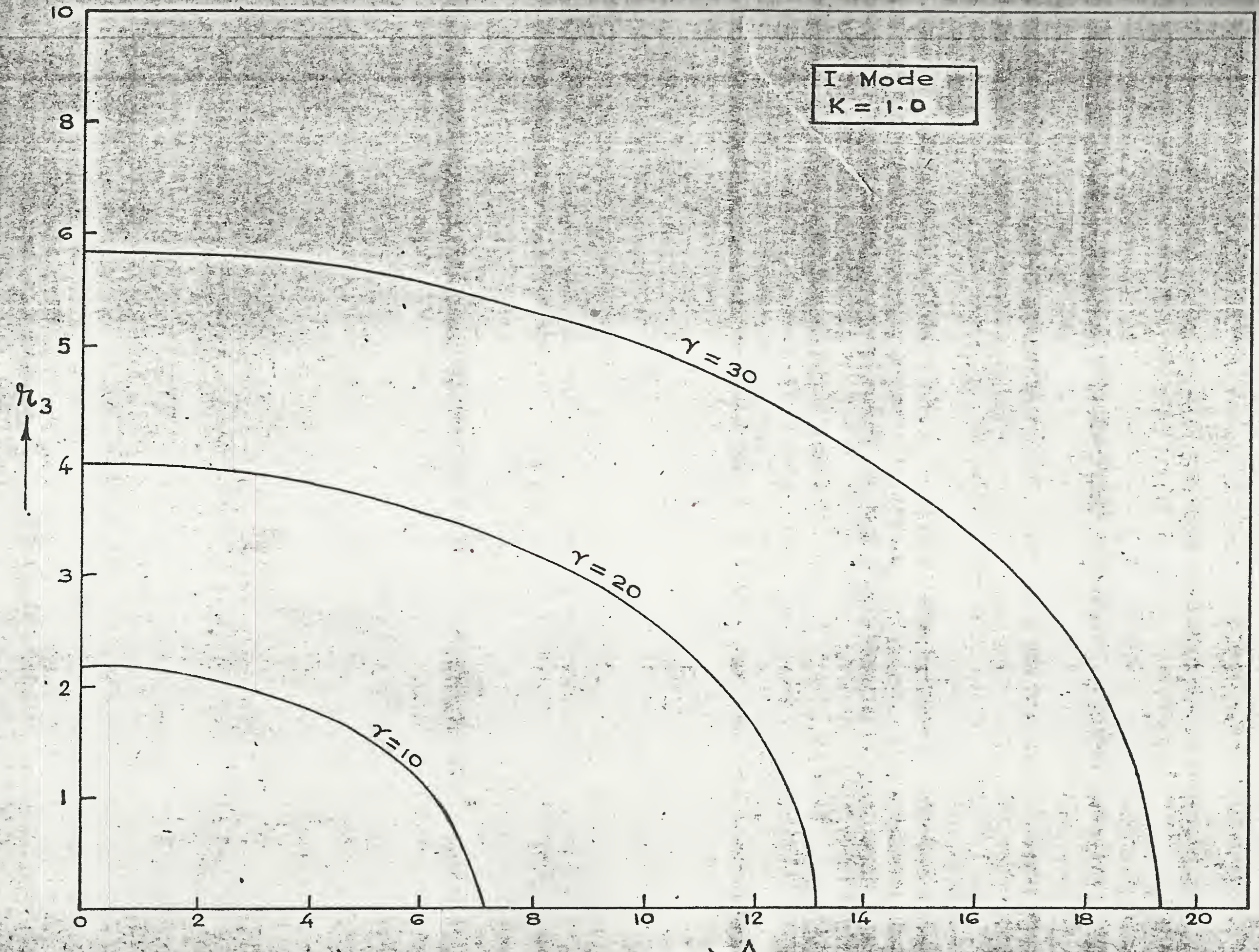



Fig. 2.5. Variation of $r_3$ with Δ and γ, for K = 1·0 for the first fundamental mode of Vibration.

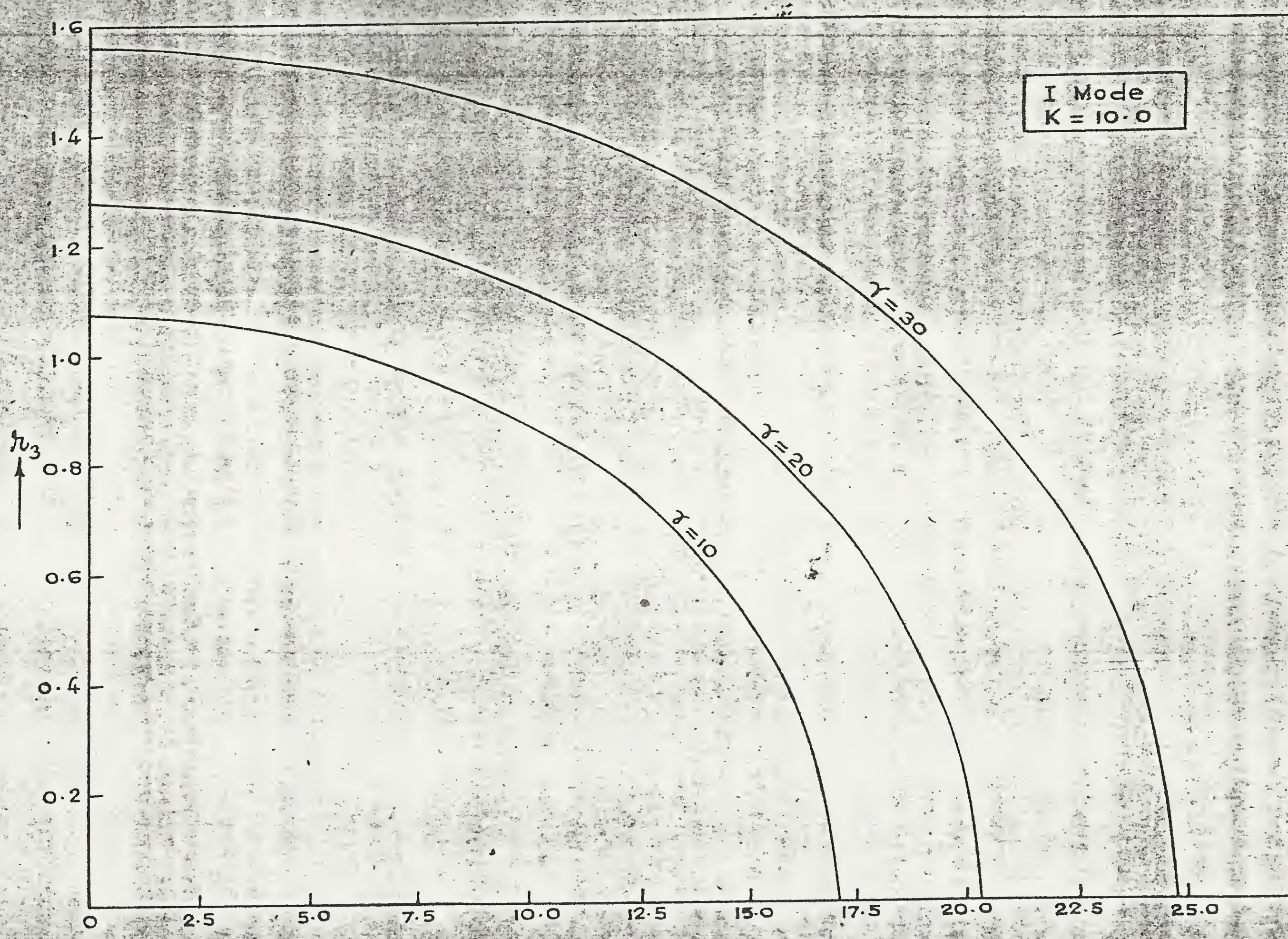



Fig. 2·6. Variation of $r_3$ with Δ and γ, for K = 10·0 for the first fundamental mode of Vibration.

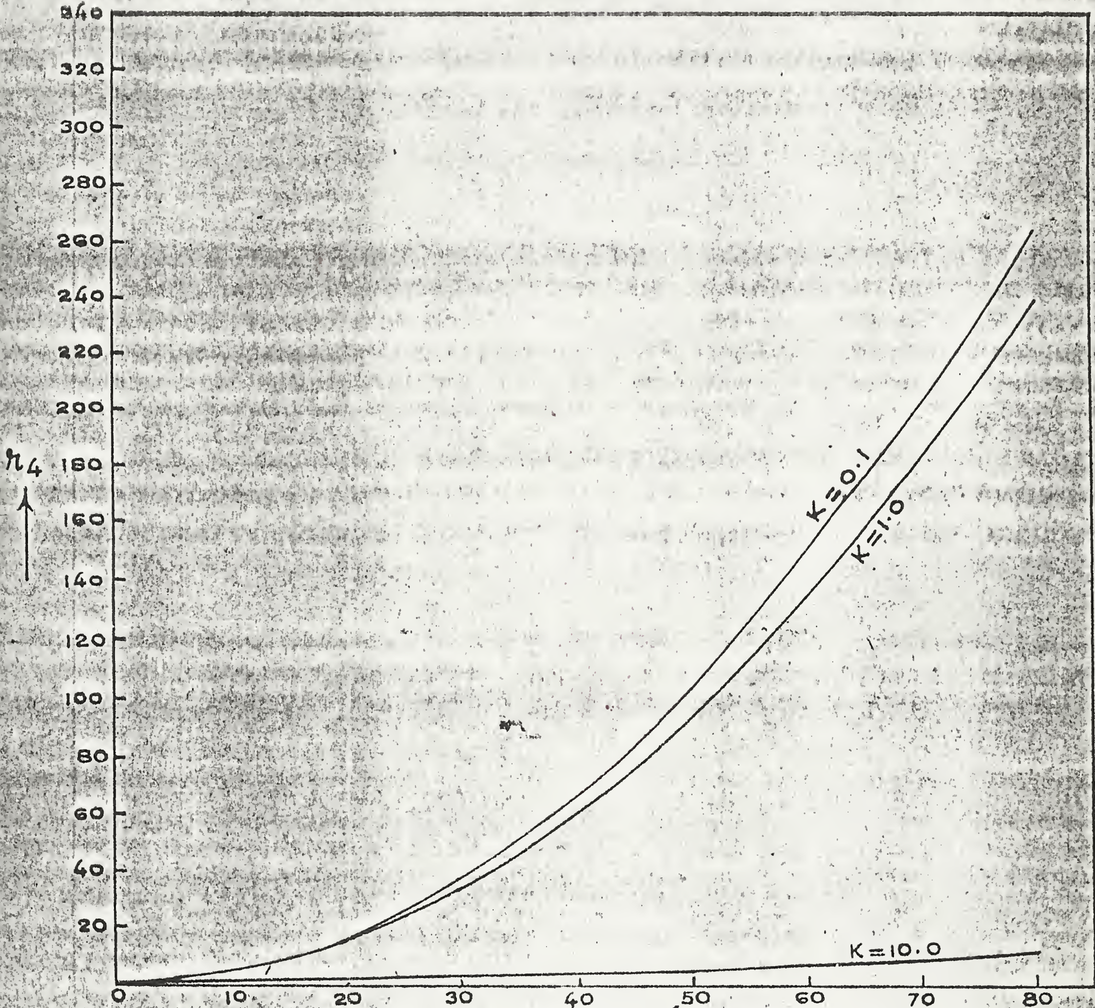



Fig. 2.7. Variation of $r_4$ with $\gamma$ for Values of K=0.1, 1.0 and 10.0.

From the above Eq.(2.56) and Fig.2.7, which shows the variation of $r_4$ with $\gamma$ for values of K = 0.1, 1.0 and 10.0, it can be observed that in the case of torsional buckling also the effect of elastic foundation is to increase the buckling load, as the elastic foundation increases the stiffness of the member against rotation. The influence of the warping parameter K is also to increase the buckling load. But relatively, the effect of warping parameter is more pronounced than that of elastic foundation.

2.72 FIXED-FIXED BEAM:

In the case of a beam which is built-in rigidly at both ends, the boundary conditions are:

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 0$$

and

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 1$$

Applying the boundary conditions to the general solutions, Eq.(2.32), frequency equation can be obtained as,

$$2-2 \cosh \alpha_1 \cos \beta_1 + \frac{(\alpha_1^2 - \beta_1^2)}{\alpha_1 \beta_1} \sinh \alpha_1 \sin \beta_1 = 0 \qquad (2.57)$$

The modal function then becomes,

$$X = D_1(\cosh \alpha_1 Z + \beta_1 \eta_1 \sinh \alpha_1 Z - \cos \beta_1 Z - \alpha_1 \eta_1 \sin \beta_1 Z) \qquad (2.58)$$

where

$$\eta_1 = \frac{\cos \beta_1 - \cosh \alpha_1}{\beta_1 \sinh \alpha_1 - \alpha_1 \sin \beta_1} = \frac{\beta_1 \sin \beta_1 + \alpha_1 \sinh \alpha_1}{\alpha_1 \beta_1 (\cos \beta_1 - \cosh \alpha_1)} \qquad (2.59)$$

2.7.3. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

With the end Z = 0, taken as the simply supported end, and the end Z = 1 as the built-in end, the boundary conditions are:

$$X = \frac{d^2X}{dZ^2} = 0 \quad \text{at } Z = 0,$$

and

$$X = \frac{dX}{dZ} = 0 \quad \text{at } Z = 1.$$

The frequency equation in this case becomes

$$\beta_1 \tanh \alpha_1 - \alpha_1 \tan \beta_1 = 0 \tag{2.60}$$

The modal function then is

$$X = D_2(\sinh \alpha_1 Z - \eta_2 \sin \beta_1 Z) \tag{2.61}$$

where

$$\eta_2 = \frac{\sinh \alpha_1}{\sin \beta_1} = \frac{\alpha_1 \cosh \alpha_1}{\beta_1 \cos \beta_1} \tag{2.62}$$

2.7.4. CANTILEVER BEAM WITH WARPING RESTRAINED:

For a cantilever beam built-in rigidly at the end Z=0 so that warping is completely prevented, and with a free end Z at Z = 1, the boundary conditions are:

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 0$$

and

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 1$$

The frequency equation for this beam can be obtained as:

$$2 + \frac{\alpha_1^4 + \beta_1^4}{\alpha_1^2\ \beta_1^2} \cosh \alpha_1 \cos \beta_1 + \frac{\alpha_1^2 - \beta_1^2}{\alpha_1\ \beta_1} \sinh \alpha_1 \sin \beta_1 = 0 \tag{2.63}$$

The modal function then becomes,

$$X = D_1(\cosh \alpha_1 Z + \beta_1 \eta_3 \sinh \alpha_1 Z - \cos \beta_1 Z - \alpha_1 \eta_3 \sin \beta_1 Z) \tag{2.64}$$

where

$$\eta_3 = \frac{\alpha_1 \sin \beta_1 - \beta_1 \sinh \alpha_1}{\alpha_1^2 \cos \beta_1 + \beta_1^2 \cos \alpha_1} =$$

$$= - \frac{\beta_1^2 \cos \beta_1 + \alpha_1^2 \cosh \alpha_1}{\alpha_1 \beta_1 (\beta_1 \sin \beta_1 + \alpha_1 \sinh \alpha_1} \tag{2.65}$$

2.7.5. CANTILEVER BEAM WITH UNRESTRAINED WARPING:

In the previous case, a cantilever beam was considered in which the supported end was fixed and offered complete restraint against warping. A cantilever beam may also be supported in a manner such that warping is free to occur at the supported end. An example is a cantilever beam supported by the ordinary framing angles and moment resistant connections used in building construction. With regard to torsion, such a support offers restraint against rotation but not warping and hence is a simple support. It is, of course, a fixed support with regard to bending.

Thus, for a cantilever simply supported at one end and free at the other, the boundary conditions are:

$$X = \frac{d^2X}{dZ^2} = 0 \quad \text{at } Z = 0\,,$$

and

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 1$$

Applying the above boundary conditions, the frequency equation can be obtained as,

$$\alpha_1^3 \tanh \alpha_1 - \beta_1^3 \tan \beta_1 = 0 \tag{2.66}$$

The modal function in this case becomes,

$$X = D_2(\sinh \alpha_1 Z + \eta_4 \sin \beta_1 Z) \tag{2.67}$$

where

$$\eta_4 = \frac{\alpha_1^2 \sinh \alpha_1}{\beta_1^2 \sin \beta_1} = \frac{\beta_1 \cosh \alpha_1}{\alpha_1 \cos \beta_1} \tag{2.68}$$

2.7.6. BEAM WITH FREE ENDS:

In the case of a beam which is free at both ends, the boundary conditions are:

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 0$$

and

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 1$$

The frequency equation for this case becomes,

$$2 - 2\cosh \alpha_1 \cos \beta_1 + \frac{\beta_1^6 - \alpha_1^6}{\alpha_1^3\,\beta_1^3} \sinh \alpha_1 \sin \beta_1 = 0 \tag{2.69}$$

The modal function therefore becomes

$$X = D_1(\cosh \alpha_1 Z + \eta_5 \sinh \alpha_1 Z + (\alpha_1/\beta_1)^2 \cos \beta_1 Z + (\beta_1/\alpha_1)\eta_5 \sin \beta_1 Z) \qquad (2.70)$$

where

$$\eta_5 = \frac{\alpha_1^3(\cos \beta_1 - \cosh \alpha_1)}{\alpha_1^3 \sinh \alpha_1 - \beta_1^3 \sin \beta)} = \frac{\beta_1^3 \sinh \alpha_1 + \alpha_1^3 \sin \beta_1}{\beta_1^3(\cos \beta_1 - \cosh \alpha_1)} \qquad (2.71)$$

## 2.8. RESULTS AND DISCUSSION:

The frequency equations derived in this section for various combinations of boundary conditions are highly transcendental in nature and can be solved only by lengthy trial-and-error procedure. As is stated earlier the same frequency equations can be used to obtain the Elastic Torsional Buckling loads for various end condition but with the only difference that for $\alpha_1$ and $\beta_1$, Equations (2.38) and (2.39) are to be used in conjunction with Equations (2.40), (2.41) and the corresponding frequency Equation. A computer program has been written in Fortran IV for solution of the above Frequency equations on IBM-1130 computer at the Computer Center, Andhra University, Waltair. Typical results for simply supported, fixed-fixed beam and beam fixed at one end and simply supported at the other for the fundamental mode (n=1) for values of K=1 and 10 are presented in Figs. 2.8 to 2.13 showing the combined influence of axial load ($\Delta$) and Elastic foundation ($\gamma$). The individual influences also can be easily observed from these graphs. Figs 2.8 and 2.9 show the variation of the fundamental torsional frequency parameter $\lambda_n^2$(n=1), for a simply supported beam,

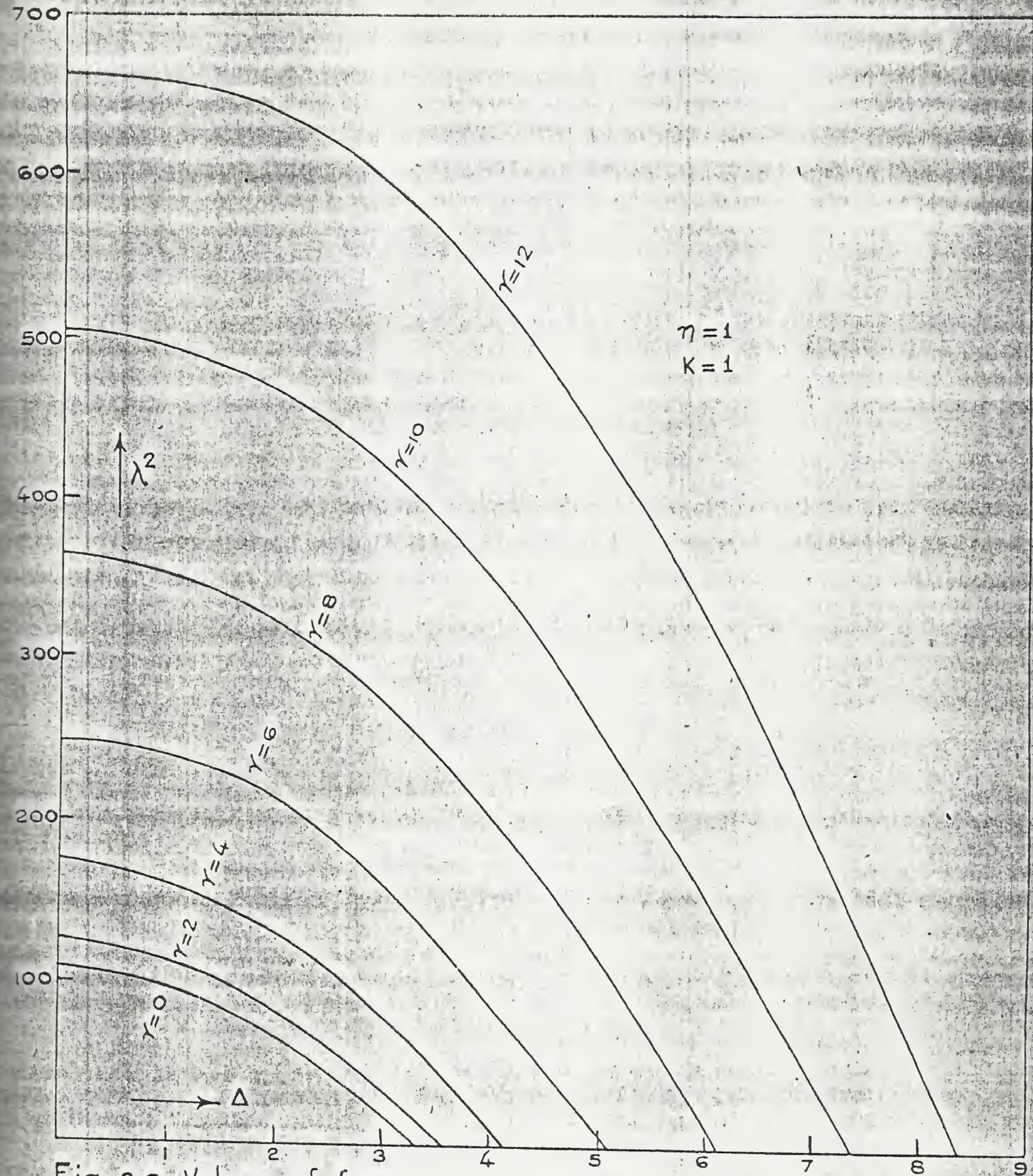



Fig. 2·8. Values of frequency & critical buckling load parameters for a simple supported beam.

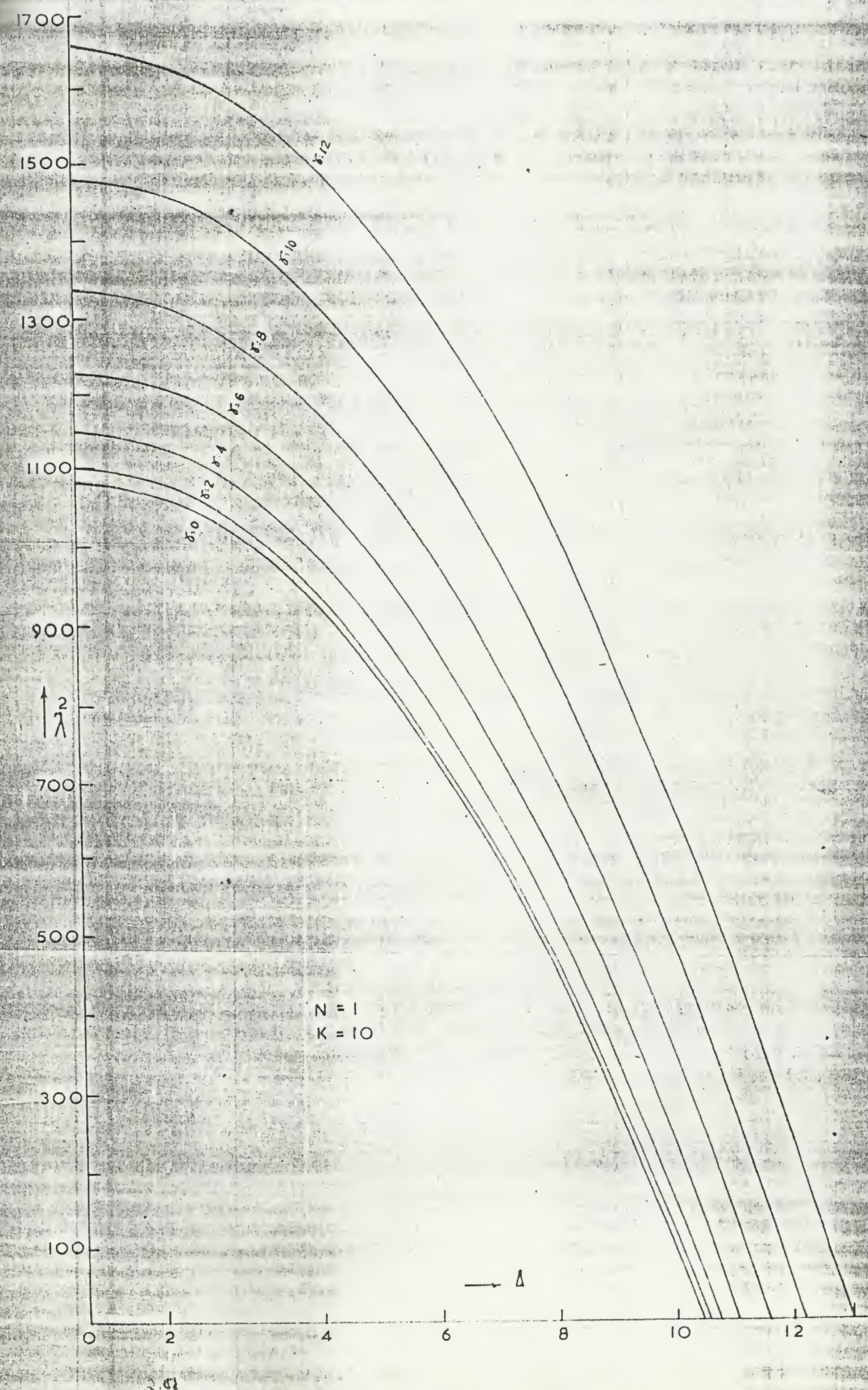



FIG. 2.9. VALUES OF FREQUENCY & CRITICAL BUCKLING PARAMETER S FOR A SIMPLY SUPPORTING BEAM.

with various values of load parameter $\Delta$ and foundation parameter $\gamma$ for values of K = 1 and 10 respectively. Figs.2.10 and 2.11 show the results for fixed-fixed beam and, the results corresponding to a beam fixed at one end and simply supported at the other are shown in Figs.2.12 and 2.13.

It can be observed from these graphs that the values of the critical buckling loads for various values of $\gamma$ can be obtained from the graphs for $\lambda^2 = 0$ ie., from the axis on which is taken. For no ~~When the~~ axial load ~~is not existing~~ the values of the frequency parameter $\lambda^2$ can be obtained from these graphs for $\Delta = 0$ ie., from the vertical axis on which $\lambda^2$ is plotted for various values of $\gamma$. The combined influence of the foundation parameter $\gamma$ and the load parameter $\Delta$ can be observed from the graphs to be due to the interaction between the individual influences on the frequency of vibration, which are interestingly opposite in nature. Independently as the load parameter increases the frequency parameter decreases to zero. In the absence of axial load, the frequency increases for increasing values of $\gamma$. It can be therefore concluded that the combined influence of foundation and load parameters is the algebraic sum of the individual influences on the frequency of vibration.

## 2.9. APPROXIMATE SOLUTIONS BY GALERKIN'S TECHNIQUE:

Except for the simply-supported beam, the frequency equations for other boundary conditions derived in the above sections (2.7) and (2.8) can be observed to be highly transcendental

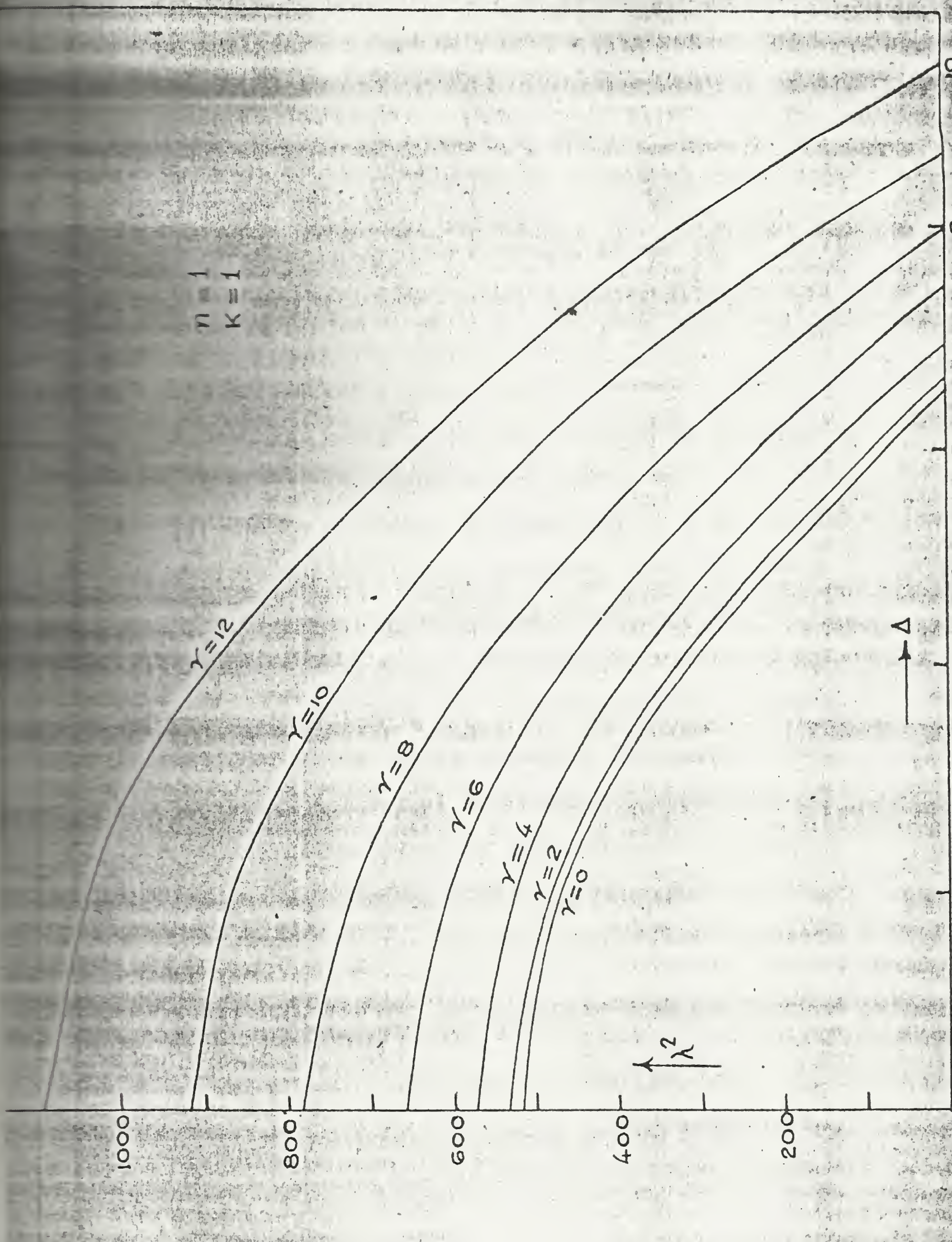



Fig 2.10: Values of frequency & critical buckling load parameters

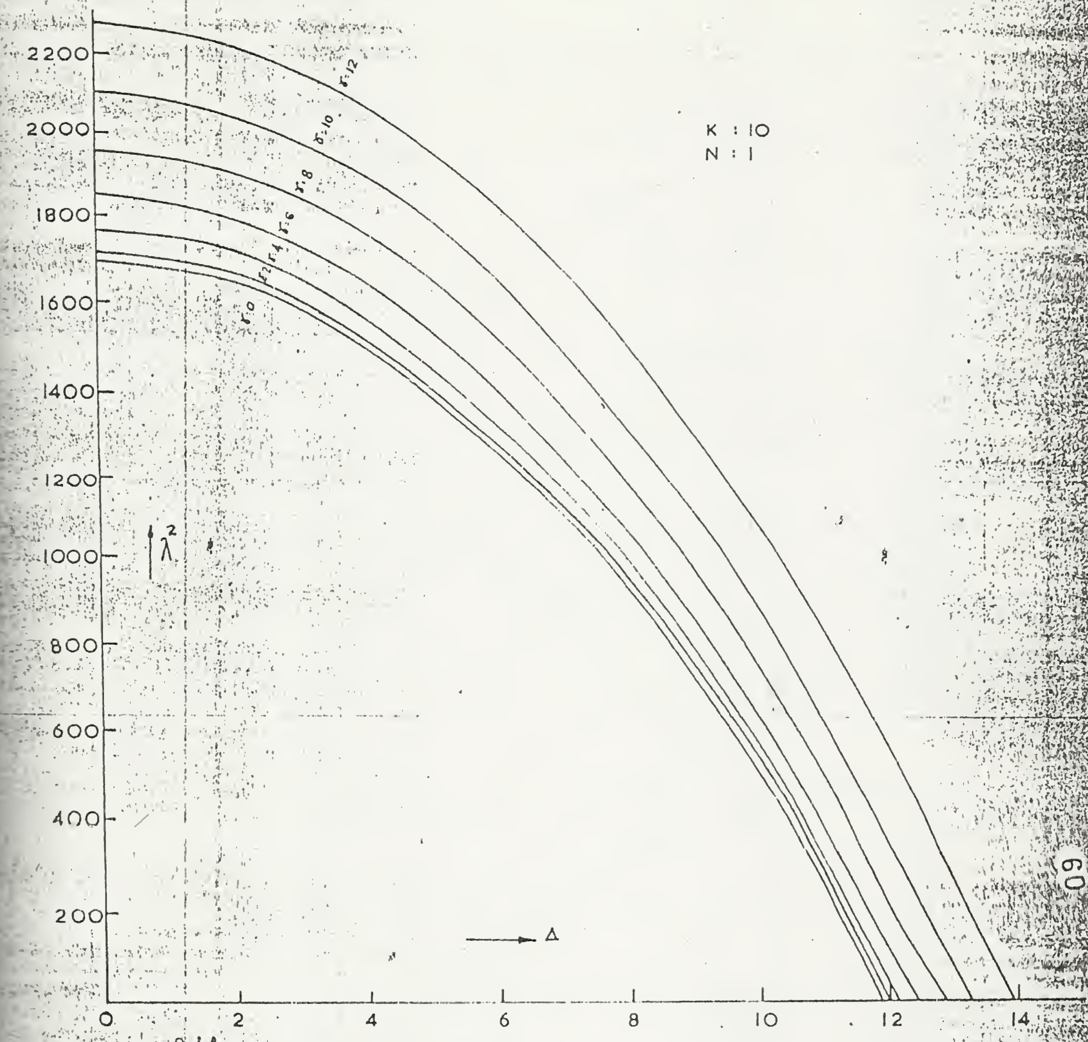



FIG.2.11. VALUES OF FREQUENCY & CRITICAL BUCKLING PARAMETERS FOR A FIXED_FIXED BEAM.

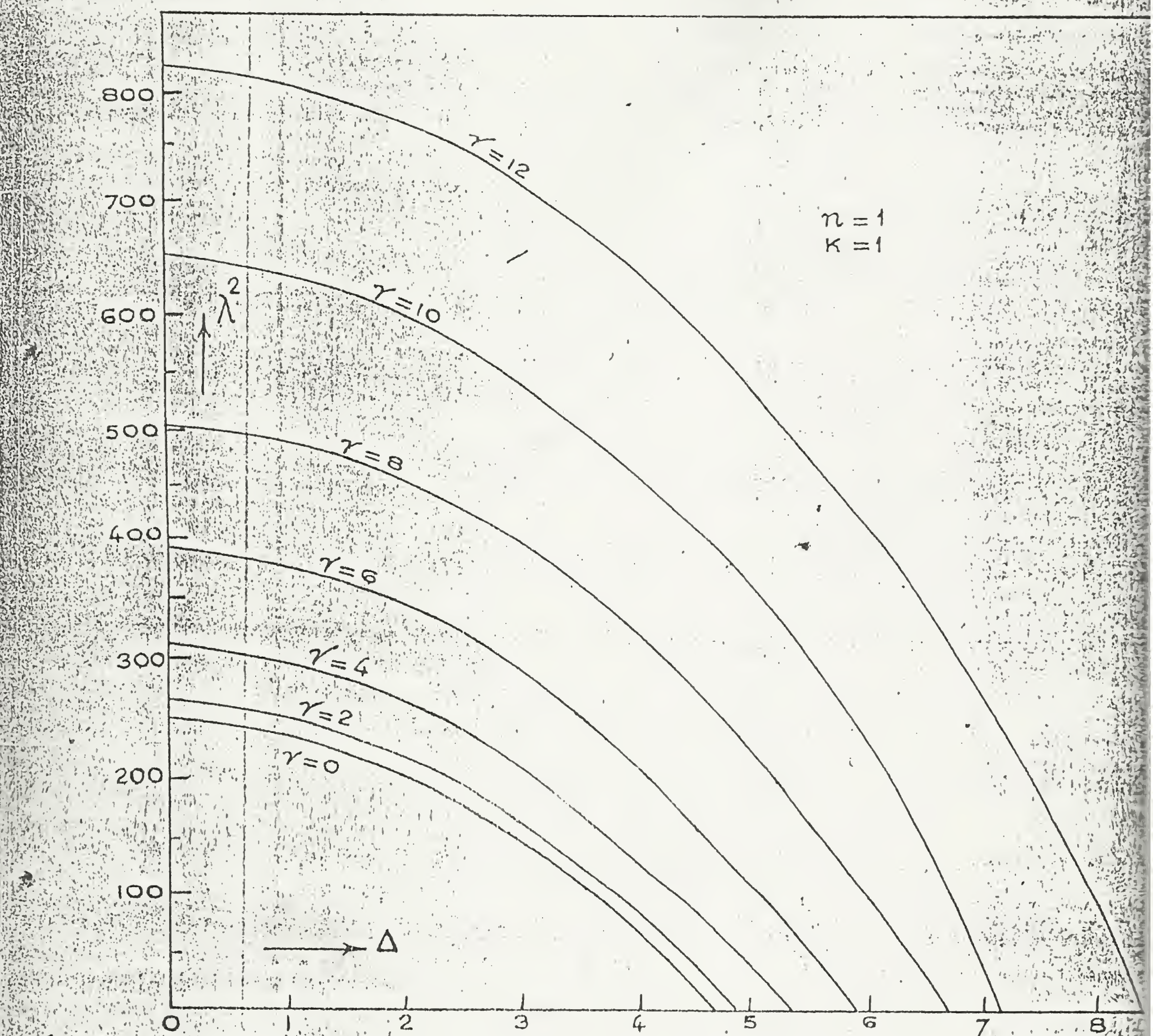



Fig. Values of frequency & critical buckling loads for a fi
simply supported beam.

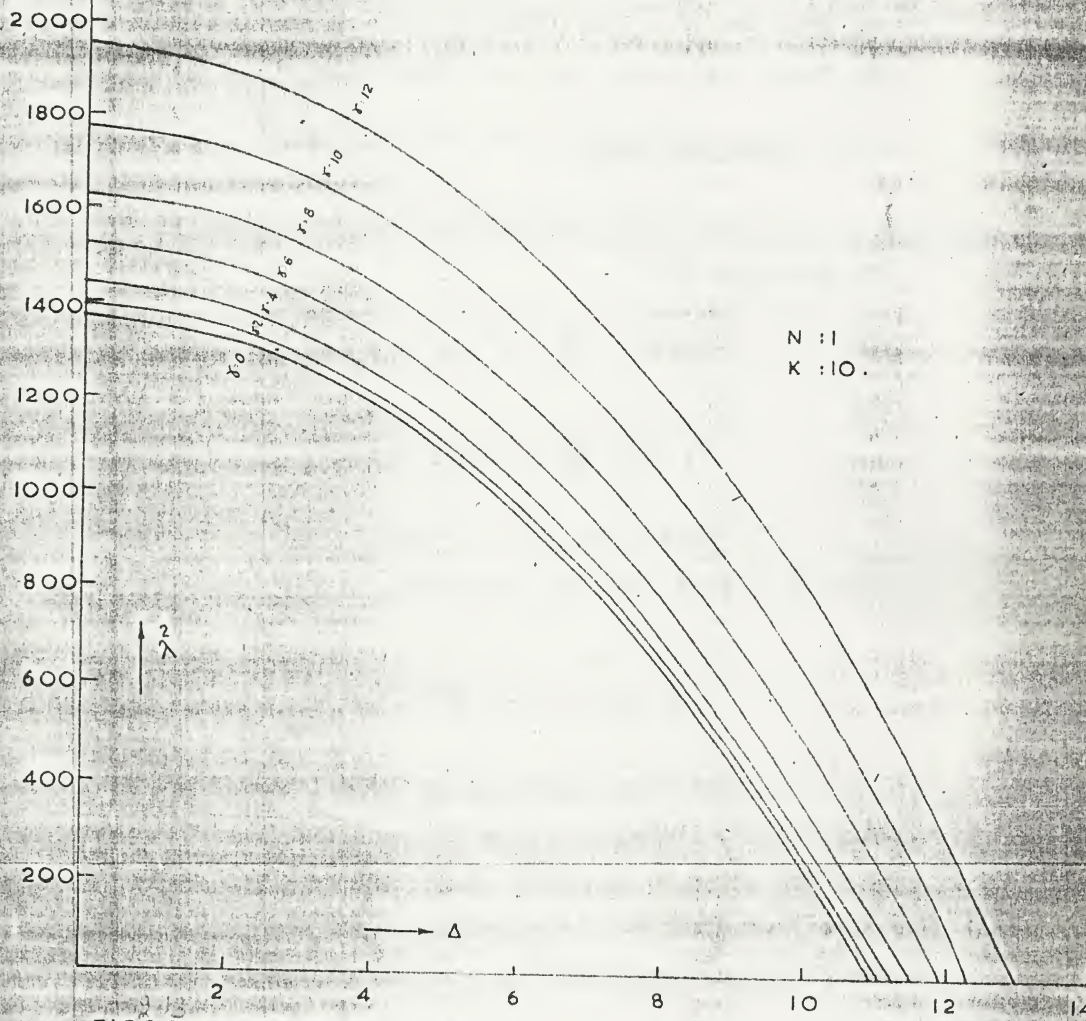



FIG.2.13 VALUES OF FREQUENCY & CRITICAL BUCKLING PARAMETERS FOR A FIXED-SIMPLY SUPPORTING BEAM.

and are solved on a digital computer only by lengthy-trial and error method. An attempt has been made in this section to derive approximate expressions for the torsional frequencies of fixed end beam and of a beam fixed at one end and simply supported at the other, utilizing the well known Galerkin's technique(77).

2.9.1. FIXED END BEAM:

The boundary conditions for a beam fixed at both ends, Z=0 and Z=1 are given by

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 0$$

and

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 1$$

To satisfy the above boundary conditions, the normal function X in this case can be assumed in the form

$$X = \sum_{n=1}^{\infty} B_n(1 - \cos 2n\pi Z) \qquad (2.72)$$

Substituting Equation (2.72) in the differential equation (2.26), orthagonalizing the resulting error with the assumed function given by Equation (2.72) and integrating the obtained expression over the whole length of the beam, the expression for the frequency parameter $\lambda$, can be obtained as,

$$\lambda = 2 \left| (n^2\pi^2/3)(4n^2\pi^2 + K^2 - \gamma^2) + \Delta^2 \right|^{1/2} \qquad (2.73)$$

In arriving Equation (2.73), only one term of the infinite series of Equation (2.72) is utilized. Hence, Equation(2.73) gives an upper bound for the natural frequency parameter .

By putting $\lambda = 0$, and $n = 1$, in Equation (2.73) the expression for the buckling load parameter $\Delta_{cr}^2$, for the fixed end beam can be obtained as

$$\Delta^2_{cr} = 4\pi^2 + K^2 + (3/\pi^2)\gamma^2 \qquad (2.74)$$

2.9.2. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

The boundary conditions in this case are:

$X = \frac{dX}{dZ} = 0$ at $Z = 0$

and

$X = \frac{d^2X}{dZ^2} = 0$ at $Z = 1$

The normal function satisfying the above boundary conditions can be assumed in the form

$$X = \sum_{n=1}^{\infty} C_n \left(\cos\frac{n\pi}{2} Z - \cos\frac{3n\pi}{2} Z\right) \qquad (2.75)$$

Substituting Equation (2.75) in the differential Equation (2.26), orthagonalizing the resulting error with the assumed function given by Equation (2.75) and integrating the obtained expression over the whole length of the beam, the equation for the frequency parameter $\lambda$ can be obtained as,

$$\lambda = \left|1.25\, n^2\pi^2 (2.05\, n^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2\right|^{1/2} \qquad (2.76)$$

Equation (2.76) also gives an upper bound for the natural torsional frequency parameter as only one term of the infinite series of

Equation (2.75) is utilized in obtaining the solution.

By putting $\lambda = 0$ and $n = 1$, in Equation (2.76), the expression for the buckling load parameter $\Delta_{cr}^2$, for the beam fixed at one end and simply supported at the other can be obtained as

$$\Delta_{cr}^2 = 2.05\,\pi^2 + K^2 + (3.2/\pi^2)\,\gamma^2 \qquad (2.77)$$

Tables 2.1 and 2.2 show the comparison between the exact results (obtained by digital computer) and the approximate results (obtained by Galerkin's technique) of the frequency parameter $\lambda$ for the first mode of vibration (n=1) of, fixed end beam and a beam fixed at one end and simply supported at the other respectively. The agreement between the results is quite good.

2.9.3. <u>LIMITTING CONDITIONS</u>:

The limiting conditions at which the combined influence of the axial compressive load and elastic foundation on the torsional frequency becomes zero, for some cases are as follows:

1) <u>Simply-Supported Beam</u>: From Equation (2.47) the limitting condition in this case becomes,

$$\gamma = 0.5\,n\pi\,\Delta \qquad (2.78)$$

2) <u>Fixed-End Beam</u>: From Equation (2.73) the limiting condition in this case is

$$\gamma = 0.574\,n\pi\,\Delta \qquad (2.79)$$

## T A B L E - 2.1

Comparison between exact and approximate values of $\lambda^2$ for the first mode of vibration of fixed-fixed beam.

| K | $\Delta$ | Values of $\lambda^2$ from exact and Approximate Analyses | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\gamma = 4.0$ | | $\gamma = 8.0$ | | $\gamma = 12.0$ | |
| | | Exact | Approximate* | Exact | Approximate* | Exact | Approximate |
| 1.0 | 0.0 | 579.792 | 596.679 | 771.995 | 788.746 | 1091.892 | 1108.556 |
| | 2.0 | 531.986 | 544.041 | 723.985 | 736.136 | 1043.794 | 1056.895 |
| | 4.0 | 387.994 | 386.127 | 579.974 | 578.268 | 899.993 | 898.235 |
| 10.0 | 0.0 | 1767.992 | 1899.473 | 1959.986 | 2091.587 | 2279.795 | 2411.562 |
| | 4.0 | 1575.874 | 1688.920 | 1767.975 | 1880.895 | 2087.992 | 2200.896 |
| | 8.0 | 999.896 | 1057.263 | 1191.982 | 1249.352 | 1511.977 | 1569.365 |

* Results from Galerkin's Technique, Eq.2.73

# T A B L E - 2.2

Comparison between exact and approximate values of $\lambda^2$ for the first mode of vibration of a fixed simply supported beam.

| K | $\Delta$ | Values of $\lambda^2$ from Exact and Approximate Analyses* | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\gamma$ = 4.0 | | $\gamma$ = 8.0 | | $\gamma$ = 12.0 | |
| | | Exact | Approximate | Exact | Approximate | Exact | Approximate |
| 1.0 | 0.0 | 314.265 | 325.950 | 506.302 | 517.894 | 826.253 | 837.892 |
| | 2.0 | 268.632 | 276.602 | 460.548 | 468.596 | 780.735 | 788.735 |
| | 4.0 | 132.226 | 128.557 | 324.676 | 320.624 | 644.378 | 640.655 |
| 10.0 | 0.0 | 1439.762 | 1547.319 | 1631.753 | 1739.526 | 1951.865 | 2059.296 |
| | 4.0 | 1257.879 | 1349.926 | 1449.536 | 1541.898 | 1769.758 | 1861.886 |
| | 8.0 | 712.010 | 757.747 | 904.893 | 949.692 | 1224.926 | 1269.686 |

* Results from Galerkin's Technique, Eq.2.76

3) <u>Beam fixed at one end and Simply Supported at the other</u>:

From Equation (2.76) the limiting condition for this case can be obtained as

$$\gamma = 0.559\, n\pi \Delta. \qquad (2.80)$$

For the above relations in various cases between $\gamma$ and $\Delta$, it is really interesting to note that there will be no influence of these two effects on the torsional frequency of vibration. This is because of the opposite nature of their individual effects and these individual effects get nullified at these limiting conditions for various cases.

## 2.10. REMARKS:

It must be recalled here that the analysis presented in this chapter neglects the effects of longitudinal inertia and shear deformation which are of importance if the effects of cross sectional dimensions on frequencies of vibration are desired. Hence, this analysis is valid for lengthy beams, ie., for beams whose cross sectional dimensions are quite small compared to the length. These second order effects such as longitudinal inertia and shear deformation, therefore, profoundly influence, the frequencies of torsional vibration at higher modes and the propagation of short wave length waves. These effects are taken into consideration in the analyses presented in the ~~coming~~ following chapters.

# CHAPTER - III

## FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS AND STABILITY OF LENGTHY THIN-WALLED BEAMS ON ELASTIC FOUNDATION*.

### 3.1. INTRODUCTION:

In Chapter II the title problem is fully analyzed from a purely mathematical approach. This approach provided us with exact solutions for the problem. One short-coming of such an approach is that due to the complex nature of the equation of motion such mathematical difficulties as non-uniform members, complex loadings, or arbitrary boundary conditions can not be easily handled.

To complement the exact solutions given in the previous Chapter, this Chapter intends to provide a means of obtaining approximate solutions to our present problem. The technique used to obtain the approximate results is the method of ''finite'' or ''discrete'' elements. Basically, the finite element method is an extension of the well known Rayleigh-Ritz method in which assumed displacement patterns are specified for an entire structure. In the finite element technique, the continuous system is replaced by a substitute system consisting of a number of finite elements linked together. Once the properties: stiffness, mass and

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* Part of the results from this Chapter were published by the author, B.V.R.Gupta and D.L.N.Rao in the Proceedings of the International Conference on Finite Element Methods in Engineering, held at Coimbatore Institute of Technology, Coimbatore, India, during 6-7 December 1974. See Ref.(48).

loading of the individual elements have been defined, the equilibrium of the substitute system can be described by a large number of equations, readily solvable on a digital computer.

Many of the early advances in the finite element method were presented in technical Journals, but recently two texts have appeared that summarized this modern technique (93, 115). These texts cover such varied topics as plane stress, plane strain, axisymmetric stress analysis, three-dimensional stress analysis, bending of beams and structural stability. To date the finite element method has been used to predict the buckling loads of trusses, beams, plates and shells. In applying the finite element method to these problems in elastic stability it has become necessary to derive the so-called ''initial stress'' or ''stability coefficient'' matrices that account for the in-plane stresses due to in-plane loads.

For problems involving large displacements the stability coefficient matrix has been termed as the ''geometric stiffness'' matrix since it accounts for the influence of large displacements on the equations of equilibrium. Using the conventional elastic stiffness matrix that accounts for the elastic bending stresses, the stability coefficient matrix for small displacements, and the mass matrix that accounts for the inertial loads, a matrix eigenvalue problem is established from which the natural frequencies, critical loads and mode shapes can be determined.

Many investigators used the above technique to predict the buckling loads of trusses (99), beams (68), plates and shells (64,89). Very recently, Pardoen (90) analyzed static and dynamic buckling of thin-walled columns using finite elements and, Barsoum ( 6 ) presented a finite element formulation for the general stability analysis of thin-walled members. The method has yet to be extended to the analysis of torsional vibrations and stability of lengthy and short thin-walled beams of open section resting on continuous winkler type elastic foundation.

Thus, a primary objective of this Chapter is to develop, for a lengthy thin-walled beam resting on Winkler type elastic foundation and subjected to an axial time-invariant compressive load, the appropriate stiffness, stability coefficient and, mass matrices necessary for a discrete element torsional vibration and stability analysis. Further, to establish the reliability of the method, the approximate finite element results will be compared with the exact solutions obtained Chapter II.

3.2. FINITE ELEMENT CONCEPT:

The use of finite elements to solve complex problems in structural mechanics has been well documented (115). The method has gained acceptance not only because of its versatility in handling complex structural problems, but also because of the highly systematic manner in which the problem is formulated and subsequently solved. Essentially, the finite element method consists of replacing th actual continuum by a mathematical model composed of structural elements of finite size having known elastic and inertial properties. These structural elements serve as building blocks of the system which, when assembled, provide approximations to the static and dynamic properties of the actual system.

The basic approach in analyzing a thin-walled beam as a net work of discrete elements can be summarized in four steps (26) as follows:

(1) The continuum must be separated by a series of lines or surfaces into a number of ''finite elements''. For a prismatic thin-walled member such as a thin-walled beam, each finite element is represented by a longitudinal segment of the whole beam.

(2) All elements are assumed to be interconnected at a discrete number of boundaries to atleast one adjacent finite element. At each of the connection boundaries a nodal point is designated. For a thin-walled beam the nodal point at the connection boundary is the shear center with generalized displacements such as translations or rotations at this point comprising the basic unknowns of the problem.

(3) The most important step in formulating the finite element procedure is choosing a function or functions to define uniquely the state of displacements within each finite element in terms of its nodal displacements.

(4) Finally, once the displacement function has been determined for the element in terms of nodal displacements, the strain

state within each element can readily be found. Typically, for elastic materials, a differential relationship exists between the displacement and strain states. The strains, together with the appropriate constitutive relation, establish the stress state within the element, the strain energy, potential energy and kinetic energy can be expressed in terms of its generalized nodal displacements.

### 3.3. BASIC FINITE ELEMENT THEORY FOR VIBRATIONS:

The finite element formulation of the general structural dynamic response problem results in the Equation (26 )

$$\bar{M}\ddot{\bar{R}} + \bar{K}\bar{R} - \bar{S}\bar{R} = \bar{F} \tag{3.1}$$

In Eq.(3.1), $\bar{K}$ is the ''total stiffness matrix'' in which the coefficients $\bar{K}_{ij}$ gives the generalized force developed at point i as the result of unit generalized displacement $\bar{R}_j = 1$ imposed on point j, all other points being restrained to zero displacement. The coefficient $\bar{S}_{ij}$ of the ''total stability coefficient matrix'' $\bar{S}$ represents the external load at coordinate i which results in a generalized displacement $\bar{R}_j = 1$ at point j. The coefficient $\bar{M}_{ij}$ of the ''total mass matrix'' $\bar{M}$ represents the mass inertia load at point i developed by a unit acceleration $\ddot{\bar{R}}_j = 1$ at point j. The matrices $\bar{R}$, $\ddot{\bar{R}}$ and $\bar{F}$ are the generalized displacements, accelerations, and loads respectively.

In the finite element deformation method, the deformations of the structure are assumed to be a function of the generalized displacements. The displacements should be continuous accross boundaries of adjoining elements, continuous over the elements, and satisfy the displacement boundary conditions, but they need not satisfy the Cauchy equilibrium equations.

Using the general procedure of the finite element method, the total structure is devided into a number of elements. These elements are connected at their corner or nodal points. Considering a typical three-dimensional element N, the displacements are given by

$$\bar{u}\ (x, y, z, t) = \bar{A}\ (x, y, z)\ \bar{R}_N(t) \qquad (3.2)$$

where the elements of $\bar{u}$ are components of the displacement vector, $\bar{A}$ is a matrix whose elements are functions of the coordinates x, y, and z, and the elements of $\bar{R}_N$ are the generalized coordinates for the N th element with time-invariant magnitudes. The strains are given in terms of nodal displacements using the strain-displacement relation.

Thus,

$$\bar{\epsilon}\ (x,y,z,t) = \bar{C}\ (x,y,z)\ \bar{R}_N(t) \qquad (3.3)$$

where $\bar{C}$ is a matrix giving the strains in terms of the generalized displacements $\bar{R}_N$. Using the stress-strain relation, the strain energy can be obtained.

Thus,

$$\bar{\sigma}\ (x,y,z,t) = \bar{D}\ (x,y,z)\ \bar{\epsilon}\ (s,y,z,t) \qquad (3.4)$$

where $\bar{\sigma}$ is a matrix of stresses, and the $\bar{D}$ matrix consists of appropriate material constants.

The strain energy U is then given by

$$U = \frac{1}{2} \int_v \bar{\epsilon}^T \bar{\sigma} \, dv \tag{3.5}$$

where $\bar{\epsilon}^T$ represents the transpose of the strain matrix $\bar{\epsilon}$ and v is the volume of the beam.

Substituting Eqs.(3.3) and (3.4) in Eq.(3.5), the strain energy expression becomes,

$$U = \frac{1}{2} \int_v \bar{R}_N^T \bar{C}^T \bar{D} \bar{C} \bar{R}_N \, dv = \frac{1}{2} \bar{R}_N^T \bar{K}_N \bar{R}_N \tag{3.6}$$

where

$$K_N = \int_v \bar{C}^T \bar{D} \bar{C} \, dv , \tag{3.7}$$

and is called stiffness matrix for the N th element. Similarly the potential energy can also be written in terms of the generalized coordinates and the stability coefficient matrix $\bar{S}_N$ for the N th element can be obtained.

The kinetic energy T is given by

$$T = \frac{1}{2} \int_v \rho \dot{\bar{u}}^T \dot{\bar{u}} \, dv \tag{3.8}$$

Substituting Eq.(3.2) into Eq.(3.8) we obtain,

$$T = \frac{1}{2} \int_v \rho \dot{\bar{R}}_N^T \bar{A}^T \bar{A} \dot{\bar{R}}_N \, dv = \frac{1}{2} \dot{\bar{R}}_N^T \bar{M}_N \dot{\bar{R}}_N , \tag{3.9}$$

where

$$\bar{M}_N = \int_v \rho \bar{A}^T \bar{A} \, dv. \tag{3.10}$$

and is called the mass matrix for the Nth element. The stiffness, stability coefficient and mass matrices for the complete connected structure is obtained by addition of the component matrices. A given column of the matrix consists of a list of generalized forces at each of the nodes for unit generalized displacement of a given node. When two or more elements have a common node, forces are simply added. Thus if $\bar{K}$ is the final stiffness matrix for the whole structure, the elements of $\bar{K}$ are built as

$$\bar{K}_{ij} = \Sigma\ (\bar{K}_{ij})_N,\ N = 1,2,... \tag{3.11}$$

and similarly

$$\bar{S}_{ij} = \Sigma\ (\bar{S}_{ij})_N\ ,\ N = 1,2,... \tag{3.12}$$

$$\bar{M}_{ij} = \Sigma\ (\bar{M}_{ij})_N\ ,\ N = 1,2,... \tag{3.13}$$

Assuming that the displacements undergo harmonic oscillation, then the displacement vector $\bar{R}_N$ can be written as

$$R_N(t) = \bar{r}_N\ e^{ip_n t} \tag{3.14}$$

where $\bar{r}_N$ is a column vector of amplitudes of the generalized displacements $\bar{R}_N$ and $p_n$ is the circular frequency of oscillation. Substituting Eq.(3.14) into Eq.(3.1) gives:

$$[\bar{K} - \bar{S}]\ [\bar{r}_N] = p_n^2\ [\bar{M}]\ [\bar{r}_N] \tag{3.15}$$

Eq.(3.15) represents an algebraic eigenvalue problem. In this finite element method, the matrices $[\bar{K}]$, $[\bar{S}]$ and $[\bar{M}]$ will be usually symmetric. If the matrices are both symmetric and positive definite, all eigenvalues $p_n^2$, will be real, positive numbers.

Moreover, the eigen vectors of symmetric matrices are independent; therefore, the matrix $[\bar{r}_N]$ is nonsingular. Another useful property of symmetric matrices is that if the eigenvectors are normalized in such a way that $\{\bar{r}_i\}^T \{\bar{r}_j\} = 1$, the inverse of the modal matrix is equal to the transpose, that is the modal matrix is orthogonal.

The eigenvalue problem for large systems can be solved by numerical schemes that are either direct or iterative. The direct methods are more general and are commonly employed, although the iterative shcemes are suitable for computations when only one or a few of the eigenvalues and their corresponding eigen vectors are needed. Among the various direct approaches to be found in literature are the Jacobi, Givens, Householder and Q R method. Among the iterative techniques are the power or Stodola-Vianello method and inverse iteration. A discussion of these various methods is given in Ref.(11). In the present work, Jacobi's method is utilized in solving the eigenvalue problems.

3.4a. <u>FUNCTIONAL REPRESENTATION OF ANGLE OF TWIST</u>:

In the past the use of polynomials as displacement functions has been popular for describing the displacement within each finite element in terms of its nodal displacements. For the present, to describe the twisting behavior of the thin-walled beam a cubic polynomial is assumed to approximate the angle of twist within each finite element. The motivation for choosing a cubic polynomial is that the contribution to the strain energy due to warping (See Eq.2.2) involves a second derivative of the angle of twist. Choosing a cubic polynomial assures that there will be a non-zero contribution from the warping term whereas if the angle of twist only varied linearly there could be no contribution from the warping term as in this case the second derivative vanishes.

For each finite element of a lengthy thin-walled beam in torsion, there are two generalized nodal displacements at the j end of the ith member. These nodal displacements are:

$\phi_j$ = angle of twist at the shear center about the longitudinal z-axis;

$\phi'_j$ = rate of change of angle of twist at the shear center about z-axis;

where the subscript j denotes the generalized displacement at the j end of the ith finite element. Similar generalized nodal displacements exist at the K end of the element. The prime denotes differentiation with respect to z.

If the twist within each finite element is assumed to vary cubicly the displacement function takes the form:

$$\phi(z) = a + bz + cz^2 + dz^3 \qquad (3.16)$$

To establish a relationship between the displacements at any interior coordinate z in terms of the generalized nodal coordinates, the four arbitrary constants in the assumed displacement function must be determined. For instance, the constants a, b, c and d can be determined from the four simultaneous equations given as follows:

$$\phi(0) = \phi_j = a \qquad (3.17)$$

$$\frac{\partial\phi}{\partial z}(0) = \frac{\partial\phi_j}{\partial z} = b \qquad (3.18)$$

$$\phi(1) = \phi_K = a+bl+cl^2+dl^3 \qquad (3.19)$$

$$\frac{\partial\phi}{\partial z}(1) = \frac{\partial\phi_K}{\partial z} = b+2cl+3dl^2 \qquad (3.20)$$

where l is the length of the element which is some fraction of the total beam length L.

Once the four coefficients have been determined, the angle of twist at any coordinate z within the element in terms of the four nodal displacements $\phi_j$, $\partial\phi_j/\partial z$, $\phi_K$ and $\partial\phi_K/\partial z$ is uniquely defined, as follows:

$$\phi(z) = \left| (1-3\xi_1^2+2\xi_1^3), (z-2\xi_1 z+\xi_1^2 z), (3\xi_1^2-2\xi_1^3), (-\xi_1 z+\xi_1^2 z) \right| \left| \begin{matrix} \phi_j \\ \phi_j' \\ \phi_K \\ \phi_K' \end{matrix} \right| \qquad (3.21)$$

where $\xi_1 = z/l$ is the dimensionless length of the element of the beam.

Eq.(3.6) can be written in an abbreviated form as:

$$\emptyset(z) = \bar{A}(z)\ \bar{R}_N(t) \tag{3.22}$$

where

$$\bar{A}(z) = \left[(1-3\bar{\xi}_1^2+2\bar{\xi}_1^3),\ (z-2\bar{\xi}_1 z+\bar{\xi}_1^2 z),\ (3\bar{\xi}_1^2-2\bar{\xi}_1^3),\ (-\bar{\xi}_1 z+\bar{\xi}_1^2 z)\right] \tag{3.23}$$

and

$$\bar{R}_N = \left[\emptyset_j\ ,\ \emptyset_j'\ ,\ \emptyset_K\ ,\ \emptyset_K'\right] \tag{3.24}$$

Similar matrix relations exist for the first and second derivatives of $\emptyset$ which can be written as:

$$\emptyset'(z) = (\bar{A}(z)\ \bar{R}_N(t)\ )' = \bar{A}_1(z)\ \bar{R}_N(t) \tag{3.25}$$

$$\emptyset''(z) = (\bar{A}\ (z)\ \bar{R}_N(t)\ )' = \bar{A}_2(z)\ \bar{R}_N(t) \tag{3.26}$$

where

$$\bar{A}_1(z) = \left[\left(-6\frac{z}{l^2} + 6\frac{z^2}{l^3}\right), \left(1-4\frac{z}{l}+3\frac{z^2}{l^2}\right), \left(6\frac{z}{l^2} - 6\frac{z^2}{l^2}\right), \left(-2\frac{z}{l} + 3\frac{z^2}{l^2}\right)\right] \tag{3.27}$$

$$\bar{A}_2(z) = \left[\left(-\frac{6}{l^2} + 12\frac{z}{l^3}\right), \left(-\frac{4}{l} + 6\frac{z}{l^2}\right), \left(\frac{6}{l^2} - 12\frac{z}{l^2}\right), \left(-\frac{2}{l} + 6\frac{z}{l^2}\right)\right] \tag{3.28}$$

The generalized velocity and accelerations can also be expressed in terms of the discretized nodal velocities and accelerations. That is:

$$\dot{\emptyset}(z) = \bar{A}(z)\ \dot{\bar{R}}_N(t) \tag{3.29}$$

and

$$\ddot{\emptyset}(z) = \bar{A}(z)\ \ddot{\bar{R}}_N(t) \tag{3.30}$$

where dots denote differentiation with respect to time t.

3.4.6 FORMULATION OF ELEMENT MATRICES:

The expressions for the kinetic energy T, strain energy U and potential energy W, derived in Chapter II (See Eqs.2.3, (2.2) and (2.4) respectively) for an element of finite length l can be written as follows:

$$T = \frac{1}{2} \int_0^l \rho I_p (\dot{\phi})^2 dz \qquad (3.31)$$

$$U = \frac{1}{2} \int_0^l \left[ EC_w (\phi'')^2 + GC_s (\phi')^2 + K_t (\phi)^2 \right] dz \qquad (3.32)$$

and

$$W = \frac{1}{2} \int_0^l \frac{PI_p}{A} (\phi')^2 dz \qquad (3.33)$$

From Hamilton's principle (See Eq.(2.1) ) we have:

$$\delta I = \delta \int_{t_1}^{t_2} (T - U + W)\, dt = 0 \qquad (3.34)$$

Direct substitution of Eqs.(3.22), (3.25), (3.26), (3.29) and (3.30) into the energy expressions (3.31), (3.32) and (3.33) yields (for the Nth element):

$$\delta I_N = \delta \int_{t_1}^{t_2} \left\{ \frac{\rho I_p}{2} \int_0^l \dot{\bar{R}}_N^T \bar{A}^T \bar{A} \dot{\bar{R}}_N dz \right.$$

$$- \left[ \frac{EC_w}{2} \int_0^l \bar{R}_N^T \bar{A}_2^T \bar{A}_2 \bar{R}_N dz + \frac{GC_s}{2} \int_0^l \bar{R}_N^T \bar{A}_1^T \bar{A}_1 \bar{R}_N dz \right.$$

$$\left. \left. + \frac{K_t}{2} \int_0^l \bar{R}_N^T \bar{A}^T \bar{A} \bar{R}_N dz \right] + \frac{PI_p}{2A} \int_0^l \bar{R}_N^T \bar{A}_1^T \bar{A}_1 \bar{R}_N dz \right\} dt = 0 \qquad (3.35)$$

Eq.(3.35) can be also written more concisely as:

$$\bar{\delta I_N} = \delta \int_{t_1}^{t_2} \frac{1}{2} \Big[ (\rho I_p L)\dot{\bar{R}}_{1N}^T \, \bar{m}_N \, \dot{\bar{R}}_{1N} - (EC_w/L^3)\bar{R}_{1N}^T \, \bar{k}_N \, \bar{R}_{1N}$$

$$+ (PI_p/AL) \, \bar{R}_{1N}^T \, \bar{s}_N \, \bar{R}_{1N} \Big] dt = 0 \qquad (3.36)$$

In Eq.(3.36) the terms $(\rho I_p L)\bar{m}_N$, $(EC_w/L^3)\bar{k}_N$ and $(PI_p/AL)\bar{s}_N$ denote respectively the mass matrix $\bar{M}_N$, the stiffness matrix $\bar{K}_N$ and the stability coefficient matrix $\bar{S}_N$ of the Nth element. The matrices $\bar{m}_N$, $\bar{k}_N$, $\bar{s}_N$ and $\bar{R}_{1N}$ are given below:

$$\bar{m}_N = \frac{1}{420N^4} \begin{bmatrix} 156N^2 & & & \\ 22N & 4 & \text{Sym.} & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (3.37)$$

$$\bar{k}_N = \begin{bmatrix} 12N^2 & & & \\ 6N & 4 & \text{Sym.} & \\ -12N^2 & -6N & 12N^2 & \\ 6N & 2 & -6N & 4 \end{bmatrix}$$

$$+ \frac{K^2}{30N^2} \begin{bmatrix} 36N^2 & & & \\ 3N & 4 & \text{Sym.} & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}$$

$$+ \frac{\gamma^2}{105N^4} \begin{vmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{vmatrix} \quad (3.38)$$

$$\bar{s}_N = \frac{1}{30N^2} \begin{vmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{vmatrix} \quad (3.39)$$

and

$$\bar{R}_{1N} = |\phi_j,\ L\ \partial\phi_j/\partial Z,\ \phi_K,\ L\ \partial\phi_K/\partial Z| \quad (3.40)$$

where N denotes the number of the elements and $Z = z/L$ is the dimensionless length of the total beam.

The equations of motion for the discretized system can now be obtained by using Eq.(3.36). Taking the variation of the integral expression of Eq.(3.36) we obtain:

$$\int_{t_1}^{t_2} \Big[ \rho I_p L \delta\dot{\bar{R}}_{1N}^T\ \bar{m}_N\ \dot{\bar{R}}_{1N} - (EC_w/L^3)\delta\bar{R}_{1N}^T\ \bar{k}_N\ \bar{R}_{1N}$$

$$+ (PI_p/AL)\ \delta\bar{R}_{1N}^T\ \bar{s}_N\ \bar{R}_{1N} \Big] dt = 0 \quad (3.41)$$

which after integration by parts over the time interval gives:

$$\rho I_p L \delta\bar{R}_{1N}^T\ \bar{m}_N\ \dot{\bar{R}}_{1N} \Big|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} \delta\bar{R}_{1N}^T \Big[ \rho I_p L\ \bar{m}_N \ddot{\bar{R}}_{1N} + (EC_w/L^3)\bar{k}_N \bar{R}_{1N} - (PI_p/AL)\bar{s}_N\ \bar{R}_{1N} \Big] dt = 0 \quad (3.42)$$

The first term in Eq.(3.42) is seen to vanish in view of the assumptions made previously that the virtual displacements $\delta\bar{R}_{1N}$ are zero at the time instants $t_1$ and $t_2$. Since the virtual displacement can be arbitrary for other times then the only way in which the integral expression in Eq.(3.42) can vanish is for the terms within the brackets to equal zero. Therefore, the governing dynamic equilibrium equations for the discretized system are:

$$\rho I_p L \,\bar{\bar{m}}_N \ddot{\bar{R}}_{1N} + (EC_w/L^3)\,\bar{\bar{k}}_N \bar{R}_{1N} - (PI_p/AL)\,\bar{\bar{s}}_N \bar{R}_{1N} = \bar{0} \tag{3.43}$$

Assuming that the displacements undergo harmonic oscillation, then the displacement vector $\bar{R}_{1N}$ can be written as:

$$\bar{R}_{1N} = \bar{\bar{r}}_N \, e^{ip_n t} \tag{3.44}$$

where $\bar{r}_N$ is a column vector of torsional amplitudes of the general torsional displacements $\bar{R}_N$ and $p_n$ is the circular frequency of torsional oscillation. Substituting Eq.(3.44) into Eq.(3.43) gives:

$$\left[ \left(\frac{EC_w}{L^3}\right)\bar{\bar{k}}_N - \left(\frac{PI_p}{AL}\right)\bar{\bar{s}}_N - \rho I_p L \, p_n^2 \, \bar{\bar{m}}_N \right] \bar{r}_N \, e^{ip_n t} = \bar{0} \tag{3.45}$$

Deviding throughout by $EC_w/L^3$ and cancelling $e^{ip_n t}$, Eq.(3.45) becomes:

$$[\bar{\bar{k}}_N - \Delta^2 \bar{\bar{s}}_N][\bar{r}_N] = \lambda^2 [\bar{\bar{m}}_N][\bar{r}_N] \tag{3.46}$$

where $\Delta^2$ and $\lambda^2$ are respectively the buckling load and frequency parameters given by:

$$\Delta^2 = \frac{PI_pL^2}{AEC_w} \tag{3.47}$$

and

$$\lambda^2 = \frac{I_pL^4p_n^2}{EC_w} \tag{3.48}$$

Eq.(3.46) represents the equations of motion for an undamped freely oscillating system.

For a beam which is stationary (not vibrating), $\lambda = 0$ and Eq.(3.46) reduces to:

$$[\bar{k}_N]\ [\bar{r}_N] = \Delta^2\ [\bar{s}_N]\ [\bar{r}_N] \tag{3.49}$$

Eq.(3.49) represents the equations of motion for the torsional buckling of a beam resting on continuous elastic foundation.

## 3.5. EQUATIONS OF EQUILIBRIUM FOR THE TOTALLY ASSEMBLED BEAM:

As previously mentioned, the matrices $\bar{K}_N$, $\bar{S}_N$, $\bar{M}_N$ and $\bar{R}_N$ pertain only to the Nth finite element and are thus denoted as the element matrices. To obtain the total strain energy, potential energy and Kinetic energy of the beam as an assemblage of N finite elements, the standard finite element procedure is employed. The procedure consists of summing the contributions of each element to form overall stiffness, stability coefficient, mass and displacement matrices which reflect the total energy of the entire beam.

The variation of total energy $\delta I$ for a thin-walled beam consisting of N finite elements is

$$\bar{\delta I} = \sum_{N=1}^{N} \bar{\delta I}_N = \sum_{N=1}^{N} \frac{1}{2} \int_{t_1}^{t_2} \Big[ \rho I_p L \delta \dot{\bar{R}}^T_{1N} \bar{m}_N \dot{\bar{R}}_{1N}$$

$$- (EC_w/L^3) \bar{\delta R}^T_{1N} \bar{k}_N \bar{R}_{1N} + (PI_p/AL) \bar{\delta R}^T_{1N} \bar{s}_N \bar{R}_{1N} \Big] dt = 0 \qquad (3.50)$$

After summation and integration by parts over the time interval Eq.(3.50) becomes:

$$\rho I_p L \ \delta \bar{R}^T \bar{m} \dot{\bar{R}}_1 \Big|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} \delta \bar{R}^T_1 \Big[ \rho I_p L \ \bar{m} \ \ddot{\bar{R}}_1 + (EC_w/L^3) \bar{k} \ \bar{R}_1 - (PI_p/AL) \bar{s} \ \bar{R}_1 \Big] dt = 0 \qquad (3.51)$$

From Eq.(3.51) the equations of equilibrium for the totally assembled beam can be written as:

$$[ \bar{k} - \triangle^2 \bar{s} ] \ [ \bar{r} ] = \lambda^2 \ [ \bar{m} ] \ [ \bar{r} ] \qquad (3.52)$$

where $\bar{k}$, $\bar{s}$, $\bar{m}$ and $\bar{r}$ denote the totally assembled matrices corresponding to the element matrices $\bar{k}_N$, $\bar{s}_N$, $\bar{m}_N$ and $\bar{r}_N$ defined previously. With the two generalized displacements possible at each node and, with the bar segmented into N elements, the number of degrees of freedom is 2 (N+1).

For a beam which is stationary and not vibrating, $\lambda = 0$ and Eq.(3.52) becomes:

$$\left[\bar{k}\right]\left[\bar{r}\right] = \triangle^2 \left[\bar{s}\right]\left[\bar{r}\right] \qquad (3.53)$$

The formulation of the above matrix equilibrium equations for the totally assembled beam, Eqs.(3.52) and (3.53) include all possible degrees of freedom, both free and restrained. The displacement vector $\bar{r}$ of this overall joint equilibrium equations is comprised of both degrees of freedom, the unknowns of the problems and known support displacements or boundary conditions.

## 3.6. BOUNDARY CONDITIONS:

It should be recalled here that for the present finite element formulation, only two generalized displacements are considered at each node. Hence, to modify the total stiffness, mass and stability coefficient matrices for various combinations of end supports the following boundary conditions are to be utilized:

(a) for a ''simply supported end'', the end of the bar does not rotate but is free to warp and hence,

$$\phi = 0 \qquad (3.54)$$

(b) for a ''clamped end'', the end of the bar is built-in rigidly so that no deformation of the end cross section can take place and we have,

$$\phi = 0 \quad \text{and} \quad \phi' = 0 \qquad (3.55)$$

(c) for a ''free end'' the total matrices are to be used without any modification.

### 3.7. METHOD OF SOLUTION:

A general computer program is written in Fortran IV to suit the IBM 1130 Computer at the Computer Center, Andhra University, Waltair, in order to obtain the eigenvalues i.e., frequency parameter $\lambda^2$ and buckling load parameter $\triangle$ for various values of the foundation parameter $\gamma$ , and their associated eigen vectors for various end conditions.

The steps involved in the computation program are as follows:

1. To read in the element properties, number of elements N, and boundary conditions.
2. To form element stiffness, stability coefficient and mass matrices.
3. To assemble the total stiffness, stability coefficient and mass matrices.
4. To modify the total matrices according to the specified boundary conditions.
5. To solve the eigenvalue problem utilizing Jacobi's method.
6. To print the given element properties, boundary conditions, number of elements, eigenvalues and their associated eigen-vectors.

### 3.8. RESULTS AND CONCLUSIONS:

The values of $\lambda^2$ for the first five frequencies of torsional vibration of simply-supported beam, obtained for a division of the beam into N = 2,4 and 6 segments for values of Warping parameter K = 1 and 10, and for values of foundation parameter $\gamma$ = 2,4,6,8,10 and 12 are shown in Tables 3.1 and 3.2 respectively, which can be observed to compare well with the exact results obtained in Chapter II. The values of $\lambda^2$ for the first five torsional frequencies of simply supported beam, for a division of the beam into N = 6 segments, for values of warping parameter K = 0.01 and 0.1, for various values of $\gamma$ = 2,4, 6,8,10 and 12 are presented in Tables 3.3 and 3.4 respectively and have compared well with the exact ones.

In Tables 3.5 and 3.6 the results for free-free and fixed-fixed beams are presented respectively for a division of the beam into N = 6 segments for values of K = 0.01, 0.1, 1.0 and 10 for various values of $\gamma$ = 2,4,6,8,10 and 12. From the results presented in Tables 3.1 to 3.6, it can be observed that the frequency parameter $\lambda^2$ increases for increasing values of the foundation parameter $\gamma$. It can also be observed that as the mode number n increases (ie., for higher modes) the influence of foundation parameter $\gamma$ decreases. The influence of increasing values of the warping parameter K can be observed to be increasing the frequency parameter $\lambda^2$ irrespective of the effect of the continuous elastic foundation. It can be concluded

# T A B L E - 3.1

Values of the frequency parameter $\lambda^2$ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter $\bar{r}$ for a value of warping parameter $\bar{K} = 1$.

| $\bar{r}$ | Number of Mode | Number of Elements: 2 | 4 | 6 | Exact Results |
|---|---|---|---|---|---|
| 0 | I | --- | --- | 107.28663 | 107.443 |
| | II | | | 1600.54248 | 1600.56 |
| | III | | | 8041.36524 | 7991.74 |
| | IV | | | 25687.86724 | 25134.9 |
| | V | | | 64403.65635 | 61225.6 |
| 2 | I | 124.05284 | 123.32942 | 123.29409 | 123.443 |
| | II | 1975.99805 | 1626.36743 | 1616.54785 | 1616.56 |
| | III | 12240.98830 | 8286.18947 | 8057.38282 | 3007.74 |
| | IV | 40503.98448 | 30985.92192 | 25703.84770 | 25510.9 |
| | V | --- | 77881.84396 | 64419.64073 | 61241.6 |
| 4 | I | 172.05264 | 171.32846 | 171.28823 | 171.443 |
| | II | 2023.99731 | 1674.36645 | 1664.54345 | 1664.56 |
| | III | 12288.99026 | 8334.18362 | 8105.38184 | 8055.74 |
| | IV | 40551.96885 | 30943.92583 | 25751.87114 | 25198.0 |
| | V | --- | 77929.82833 | 64467.64854 | 61289.6 |

TABLE - 3.1 (Contd.)

| | | | | | |
|---|---|---|---|---|---|
| 6 | I | 252.05285 | 251.32824 | 251.29016 | 251.443 |
| | II | 2103.99805 | 1754.36548 | 1744.54053 | 1744.56 |
| | III | 12368.98440 | 8414.18557 | 8185.38965 | 8135.74 |
| | IV | 40631.98448 | 31023.93364 | 25831.85161 | 25278.9 |
| | V | --- | 78009.85958 | 64547.63291 | 61369.6 |
| 8 | I | 364.05273 | 363.32800 | 363.29058 | 363.443 |
| | II | 2215.99805 | 1866.36572 | 1856.55054 | 1856.56 |
| | III | 12480.99026 | 8526.18947 | 8297.37502 | 8247.74 |
| | IV | 40743.98448 | 31135.92583 | 25943.87114 | 25390.9 |
| | V | --- | 78121.84396 | 64659.64854 | 61481.6 |
| 10 | I | 508.05279 | 507.32733 | 507.28814 | 507.443 |
| | II | 2359.99756 | 2010.36572 | 2000.54663 | 2000.56 |
| | III | 12624.98635 | 8670.18947 | 8441.35939 | 8397.74 |
| | IV | 40887.87666 | 31279.92583 | 26087.83208 | 25534.9 |
| | V | --- | 78265.84396 | 64803.60948 | 61625.6 |
| 12 | I | 684.05285 | 683.32605 | 683.28711 | 683.443 |
| | II | 2535.99756 | 2186.36524 | 2176.53809 | 2176.56 |
| | III | 12800.98830 | 8846.18557 | 8617.34767 | 8567.74 |
| | IV | 41063.97666 | 31455.92583 | 26263.88677 | 25710.9 |
| | V | --- | 78441.84396 | 64779.62510 | 61801.6 |

## T A B L E - 3.2

Values of the Frequency parameter $\lambda^2$ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter $\bar{r}$ for a value of warping parameter $\bar{K} = 10$.

| $\bar{r}$ | Number of Mode | Number of Elements 2 | 4 | 6 | Exact Results |
|---|---|---|---|---|---|
| 0 | I | --- | --- | 1084.37207 | 1085.32 |
| | II | | | 5509.04395 | 5512.07 |
| | III | | | 16838.35552 | 16792.6 |
| | IV | | | 41347.17977 | 40780.9 |
| | V | | | 88955.62521 | 85672.6 |
| 2 | I | 1101.46338 | 1100.42187 | 1100.37646 | 1101.32 |
| | II | 5935.99415 | 5536.11720 | 5525.04298 | 5528.07 |
| | III | 21571.76177 | 17105.23832 | 16854.35161 | 16808.6 |
| | IV | 57135.97666 | 46735.88291 | 41363.17196 | 40796.9 |
| | V | --- | 102839.67208 | 88871.62521 | 85688.6 |
| 4 | I | 1149.46362 | 1148.42114 | 1148.37182 | 1149.32 |
| | II | 5983.99513 | 5584.11913 | 5573.03223 | 5576.07 |
| | III | 21619.76177 | 17153.24223 | 16902.34770 | 16856.6 |
| | IV | 57183.96885 | 46783.87510 | 41411.21102 | 40924.9 |
| | V | --- | 102887.67208 | 89019.65646 | 85736.6 |

T A B L E - 3.2 (Contd.)

| | | | | | |
|---|---|---|---|---|---|
| 6 | I | 1229.46362 | 1228.42090 | 1228.37012 | 1229.32 |
| | II | 6063.99317 | 5664.12306 | 5653.03907 | 5656.07 |
| | III | 21699.76177 | 17283.23832 | 16982.35552 | 16936.6 |
| | IV | 57263.96885 | 46863.88291 | 41491.14852 | 40924.9 |
| | V | --- | 102967.67208 | 89099.65646 | 85816.6 |
| 8 | I | 1341.46338 | 1340.42163 | 1340.37402 | 1341.32 |
| | II | 6175.99610 | 5776.11427 | 5765.04102 | 5768.07 |
| | III | 21811.76177 | 117345.25004 | 17094.35552 | 17048.6 |
| | IV | 57375.97666 | 46975.88291 | 41603.16415 | 41036.9 |
| | V | --- | 103079.73458 | 89211.59396 | 85928.6 |
| 10 | I | 1485.46362 | 1484.41992 | 1484.37036 | 1485.32 |
| | II | 6319.99513 | 5920.11915 | 5909.04298 | 5912.07 |
| | III | 22131.76177 | 17489.23832 | 17238.35161 | 17192.6 |
| | IV | 57519.95323 | 47119.86729 | 41747.17977 | 41180.9 |
| | V | --- | 103223.67208 | 89355.64083 | 86072.6 |
| 12 | I | 1661.46338 | 1660.42016 | 1660.36767 | 1661.32 |
| | II | 6495.99415 | 6096.11915 | 6085.04981 | 6088.07 |
| | III | 22131.75786 | 17665.23442 | 17414.33989 | 17368.6 |
| | IV | 57695.98448 | 47295.87510 | 41923.15634 | 41356.6 |
| | V | --- | 103399.65646 | 89531.64083 | 86248.6 |

TABLE - 3.3

Values of the Frequency parameter $\lambda^2$ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter r for a value of warping parameter K = 0.01.

| $\bar{r}$ | Number of Mode | Number of Elements 6 | Exact Results |
|---|---|---|---|
| | I | 97.42036 | |
| | II | 1561.06689 | |
| 0 | III | 7952.49806 | --- |
| | IV | 25529.69145 | |
| | V | 64155.57041 | |
| | I | 113.42420 | 113.566986 |
| | II | 1577.08105 | 1577.060061 |
| 2 | III | 7968.49513 | 7918.854505 |
| | IV | 25545.68363 | 24992.914115 |
| | V | 64171.59385 | 60994.773544 |
| | I | 161.41571 | 161.566986 |
| | II | 1625.07593 | 1625.060061 |
| 4 | III | 8016.49122 | 7966.854505 |
| | IV | 25593.67192 | 25040.914115 |
| | V | 64219.60948 | 61042.773544 |
| | I | 241.41577 | 241.566986 |
| | II | 1705.07324 | 1705.060061 |
| 6 | III | 8096.49122 | 8046.854505 |
| | IV | 25673.68363 | 25120.914115 |
| | V | 64299.58604 | 61122.773544 |
| | I | 353.42065 | 353.567017 |
| | II | 1817.07251 | 1817.060061 |
| 8 | III | 8208.49221 | 8159.854505 |
| | IV | 25785.68754 | 25232.914115 |
| | V | 64411.55479 | 61234.773544 |
| | I | 497.42071 | 497.567017 |
| | II | 1961.07226 | 1961.060061 |
| 10 | III | 8352.50002 | 8302.855491 |
| | IV | 25929.67582 | 25376.914115 |
| | V | 64555.60948 | 61378.773544 |
| | I | 673.41674 | 673.567018 |
| | II | 2137.07080 | 2137.060065 |
| 12 | III | 8528.49807 | 8478.855491 |
| | IV | 26105.66801 | 25552.914115 |
| | V | 64731.57823 | 61554.773544 |

T A B L E - 3.4.

Values of the Frequency parameter $\lambda^2$ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter $\gamma$ for a value of warping parameter $\bar{K}$ = 0.100.

| $\gamma$ | Number of Mode | Number of Elements 6 | Exact Results |
|---|---|---|---|
| 0 | I | 97.51748 | --- |
| | II | 1561.46094 | |
| | III | 7953.36817 | |
| | IV | 25531.24613 | |
| | V | 64158.03915 | |
| 2 | I | 113.52183 | 113.664779 |
| | II | 1577.46582 | 1577.451174 |
| | III | 7969.38282 | 7919.735364 |
| | IV | 25547.23442 | 24994.476615 |
| | V | 64174.04696 | 60997.218841 |
| 4 | I | 161.51513 | 161.664795 |
| | II | 1625.46216 | 1625.451174 |
| | III | 8017.38184 | 7967.735364 |
| | IV | 25595.25395 | 25042.437615 |
| | V | 64222.00010 | 61045.218841 |
| 6 | I | 241.51611 | 241.664795 |
| | II | 1705.46167 | 1705.451174 |
| | III | 8097.40040 | 8047.735364 |
| | IV | 25675.24613 | 25122.476615 |
| | V | 64302.00790 | 61125.218841 |
| 8 | I | 353.51928 | 353.664795 |
| | II | 1817.46264 | 1817.451174 |
| | III | 8209.38088 | 8159.735364 |
| | IV | 25787.25786 | 25234.476615 |
| | V | 64413.99220 | 61237.218841 |
| 10 | I | 497.51690 | 497.664795 |
| | II | 1961.46142 | 1961.451174 |
| | III | 8353.38479 | 8303.736354 |
| | IV | 25931.24613 | 25378.476615 |
| | V | 64558.05477 | 61381.218841 |
| 12 | I | 673.51562 | 673.664796 |
| | II | 2137.45606 | 2137.45117 |
| | III | 8529.28283 | 8479.736354 |
| | IV | 26107.25395 | 25554.476615 |
| | V | 64734.04696 | 61457.218841 |

TABLE - 3.5

Values of the Frequency parameter $\lambda^2$ for fixed-fixed thin-walled beams of open section on Elastic foundation for various values of foundation and warping parameters $\gamma$ and $\bar{K}$ respectively (N = 6).

| K | Mode No | Values of $\gamma$: 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|
| 0.01 | I | 500.82769 | 516.82995 | 564.82458 | 644.82568 | 756.82763 | 900.82885 | 1076.82617 |
| | II | 3818.47852 | 3834.48145 | 3882.47363 | 3962.47803 | 4074.46875 | 4218.48048 | 4394.46583 |
| | III | 14827.81642 | 14843.81642 | 14891.81642 | 14971.81642 | 15083.82424 | 15227.81642 | 151403.80861 |
| | IV | 41352.17196 | 41368.14852 | 41416.12509 | 41496.10946 | 41608.14071 | 41752.14071 | 41928 .14071 |
| | V | 93087.53146 | 93103.51583 | 93151.56271 | 93231.57833 | 93343.54708 | 93487.50021 | 93663.53146 |
| 0.10 | I | 500.94867 | 516.95300 | 564.94653 | 644.94665 | 756.94885 | 900.94922 | 1076.94800 |
| | II | 3818.93018 | 3834.93115 | 3882.93213 | 3962.93604 | 4074.92383 | 4218.93458 | 4394.92969 |
| | III | 14828.78713 | 14844.79299 | 14892.80080 | 14972.78713 | 15084.79689 | 15228.78713 | 15404.79494 |
| | IV | 41353.86729 | 41369.86729 | 41417.85948 | 41497.85166 | 41609.85166 | 41753.84385 | 41929.84385 |
| | V | 93090.20330 | 93106.14080 | 93154.15643 | 93234.18768 | 93346.18768 | 93490.17205 | 93666.18768 |
| 1.00 | I | 513.12353 | 529.12585 | 577.11975 | 657.12426 | 769.12646 | 913.12133 | 1089.12060 |
| | II | 3864.54248 | 3880.54199 | 3928.53418 | 4008.54492 | 4120.53712 | 4262.54102 | 4440.53419 |
| | III | 14927.01760 | 14943.01955 | 14991.00978 | 15071.02936 | 15183.01955 | 15327.01369 | 15503.01174 |
| | IV | 41525.59385 | 41541.58604 | 41589.58604 | 41669.60166 | 41781.58604 | 41925.60948 | 42101.59385 |
| | V | 93361.31271 | 93377.31271 | 93425.34396 | 93505.29708 | 93617.31271 | 93761.28146 | 93937.28146 |
| 10.0 | I | 1683.98877 | 1699.98999 | 1747.98486 | 1827.98315 | 1939.98266 | 2083.98535 | 2259.98486 |
| | II | 8360.31252 | 8376.32033 | 8424.31056 | 8504.31447 | 8616.31838 | 8760.31252 | 8936.31252 |
| | III | 24732.19145 | 24748.21488 | 24796.19926 | 24876.21098 | 24988.19926 | 25132.20317 | 25308.20707 |
| | IV | 58721.25790 | 58737.28134 | 58785.24227 | 58865.25790 | 58977.24227 | 59121.25009 | 59297.26571 |
| | V | 120469.45333 | 120485.46896 | 120533.45333 | 120613.48458 | 120725.48458 | 120869.45333 | 121045.48458 |

# T A B L E - 3.6

Values of the Frequency parameter $\lambda^2$ for free-free thin-walled beams of open section on Elastic Foundation for various values of foundation and warping parameters $\gamma$ and K respectively (N = 6).

| K | Mode No | Values of $\gamma$: 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|
| 0.010 | I | 0.00114 | 16.00515 | 63.99741 | 143.99972 | 256.00122 | 399.99932 | 576.0036 |
| | II | 0.00815 | 16.01287 | 64.00315 | 144.01077 | 257.01220 | 400.00708 | 576.01135 |
| | III | 500.80816 | 516.81799 | 564.80713 | 644.82763 | 756.81396 | 900.81250 | 1076.81518 |
| | IV | 3816.42822 | 3832.41455 | 3880.41085 | 3960.41699 | 4072.41992 | 4216.42481 | 4392.42090 |
| | V | 14785.43752 | 14801.44143 | 14849.44143 | 14929.45510 | 15041.43752 | 15185.45705 | 15361.43752 |
| 0.100 | I | 0.00094 | 16.00503 | 64.00390 | 144.00061 | 256.00116 | 399.99908 | 576.00036 |
| | II | 0.12253 | 16.12235 | 64.12205 | 144.11544 | 256.12005 | 400.13055 | 576.11731 |
| | III | 501.30493 | 517.30456 | 565.31860 | 645.31494 | 757.29931 | 901.31140 | 1077.29736 |
| | IV | 3817.47998 | 3833.48096 | 3881.46875 | 3961.50342 | 4073.48340 | 4217.49610 | 4393.47559 |
| | V | 14787.30080 | 14803.31056 | 14851.32434 | 14931.32814 | 15043.29689 | 15187.32033 | 15363.29885 |
| 1.00 | I | 0.00117 | 16.00636 | 63.09979 | 143.99942 | 255.99951 | 399.99969 | 575.99572 |
| | II | 11.95039 | 27.94924 | 75.95919 | 155.94809 | 267.95068 | 411.95587 | 587.94665 |
| | III | 550.23401 | 566.23986 | 614.22033 | 694.23193 | 806.22277 | 950.21899 | 1126.19848 |
| | IV | 13925.66797 | 13941.63721 | 3989.68360 | 4069.65088 | 4131.66602 | 4325.68653 | 4501.66407 |
| | V | 14974.07033 | 14990.07033 | 15038.07033 | 15118.06642 | 15230.08791 | 15374.05471 | 15550.04103 |
| 10.0 | I | 0.00858 | 16.00782 | 64.00463 | 144.00341 | 256.00756 | 400.00787 | 576.00903 |
| | II | 1046.32202 | 1062.31982 | 1110.32690 | 1190.31909 | 1302.32104 | 1446.32300 | 1622.32055 |
| | III | 4926.61915 | 4942.59766 | 4990.61915 | 5070.60743 | 5132.61524 | 5326.61915 | 5502.62013 |
| | IV | 14068.30471 | 14084.29689 | 14132.31252 | 14212.30471 | 14324.32228 | 14468.32619 | 14644.31056 |
| | V | 33131.40634 | 33147.40634 | 33195.40634 | 33275.39071 | 33337.41415 | 33531.39071 | 33707.40634 |

therefore, that increase in the values of warping parameter K and foundation parameter $\gamma$ contribute for the increase in the torsional frequency parameter $\lambda^2$.

In Tables 3.7, 3.8 and 3.9, the values of the frequency parameter $\lambda^2$ for the first five modes of vibration are presented for simply-supported, fixed-fixed and, fixed-simply supported beams respectively, for various values of axial load parameter $\Delta$ and foundation parameter $\gamma$, for a value of warping parameter K = 1. These results are given for a division of the beam into four and six segments. It can be observed from Table 3.7, that the results for the simply-supported beams compare well with the exact ones. It can be also noticed that increase in the value of axial load parameter $\Delta$, for any constant or zero values of the foundation parameter $\gamma$ and warping parameter K, is to decrease the value of the frequency parameter $\lambda^2$. Similarly it can be observed that, for any constant or zero values of the axial load parameter $\Delta$, the increase in the values of foundation parameter $\gamma$ and warping parameter K is to increase the value of the frequency parameter $\lambda^2$.

Hence It can be concluded that the combined influence of axial load parameter $\Delta$, foundation parameter $\gamma$ and warping parameter K on the frequency parameter $\lambda^2$ is the algebraic sum of the individual influences of these parameters. In general, for all the cases presented here, the results from the finite element analysis are in excellent agreement with the exact results from Chapter II, and the convergence of the results is quite satis-

T A B L E - 3.7

Values of the frequency parameter $\lambda^2$ for simply supported beams for various values of axial load parameter $\Delta$ and foundation parameter $\gamma$ for a value of K = 1.

| Value of $\gamma$ | Values of $\Delta$ | No.of Mode | Number of Elements 4 | Number of Elements 6 | Exact Results |
|---|---|---|---|---|---|
| 0.0 | 2.0 | I | 67.5211 | 67.6339 | 67.8010 |
| | | II | 1286.9080 | 1294.5712 | 1440.1235 |
| | | III | 7940.3071 | 7678.1423 | 7623.7256 |
| | | IV | 30635.5009 | 25771.8011 | 24463.2071 |
| | | V | 76947.9860 | 63627.9682 | 60141.0001 |
| 6.0 | 2.0 | I | 211.5192 | 21.9334 | 211.8011 |
| | | II | 1430.9046 | 1438.5728 | 1584.1235 |
| | | III | 8084.3116 | 7822.1494 | 7767.7256 |
| | | IV | 30779.5448 | 25915.8241 | 24607.2071 |
| | | V | 77092.2992 | 63772.0157 | 60285.0001 |
| 6.0 | 3.5 | I | 129.9805 | 117.5148 | 130.3764 |
| | | II | 1107.5051 | 1111.9768 | 1258.4253 |
| | | III | 7343.5527 | 7088.6543 | 7034.9043 |
| | | IV | 29453.4244 | 24607.4609 | 23304.4141 |
| | | V | 75010.2544 | | 58249.3829 |

T A B L E - 3.8

Values of the frequency parameter $\lambda^2$ for fixed-fixed beams for various values of axial load parameter $\Delta$ and foundation parameter $\gamma$ for a value of K = 1.

| Value of $\gamma$ | Value of $\Delta$ | Mode No. | Number of Elements | |
|---|---|---|---|---|
| | | | 4 | 6 |
| 0.0 | 2.0 | I | 606.6059 | 474.5637 |
| | | II | 3732.2152 | 3719.4751 |
| | | III | 14463.6348 | 14153.3851 |
| | | IV | 53954.4631 | 40699.0235 |
| | | V | 146916.5902 | 93660.2189 |
| 6.0 | 2.0 | I | 750.6064 | 625.8250 |
| | | II | 3876.2219 | 3863.4802 |
| | | III | 14607.4632 | 14297.6426 |
| | | IV | 54098.5733 | 40843.0235 |
| | | V | 147060.7452 | 93810.2189 |
| 6.0 | 6.6 | I | 266.1976 | 210.3856 |
| | | II | 2036.6875 | 2022.8406 |
| | | III | 10659.9258 | 10340.2461 |
| | | IV | 46728.1719 | 33990.2423 |
| | | V | 135125.3488 | 82926.8439 |

## TABLE - 3.9

Values of the frequency parameter $\lambda^2$ for fixed-simply supported beams for various values of axial load parameter $\Delta$ and foundation parameter $\gamma$ for a value of $K = 1$.

| Value of $\gamma$ | Value of $\Delta$ | Mode No. | Number of Elements: 4 | Number of Elements: 6 |
|---|---|---|---|---|
| 0.0 | 3.0 | I | 178.7215 | 149.8247 |
| | | II | 2069.1348 | 1931.5761 |
| | | III | 10244.8752 | 10065.7076 |
| | | IV | 38389.3486 | 30721.6524 |
| | | V | 102718.9681 | 76119.4065 |
| 6.0 | 3.0 | I | 322.7196 | 299.8219 |
| | | II | 2213.1309 | 2075.5769 |
| | | III | 10388.8594 | 10209.6878 |
| | | IV | 38533.2974 | 30865.6524 |
| | | V | 102863.2427 | 76263.4221 |
| 6.0 | 4.7 | I | 257.3521 | 210.5931 |
| | | II | 1653.0818 | 1498.0771 |
| | | III | 9167.3867 | 9004.3523 |
| | | IV | 36271.5834 | 28703.0196 |
| | | V | 99361.3712 | 72872.9689 |

factory for a division of the beam into six elements. Hence, the finite element model presented in this Chapter, which includes the effects of warping, axial compressive load and elastic foundation is quite satisfactory and yields good results.